

ON PAVLOV'S CONJECTURE ON PRESENTABLY SYMMETRIC MONOIDAL ∞ -CATEGORIES

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ABSTRACT. We prove that the ∞ -category of presentably symmetric monoidal ∞ -categories is equivalent to that of combinatorial symmetric monoidal model categories and left Quillen monoidal functors. This proves the conjecture by Pavlov.

Size issues arise inevitably in category theory, and careless handling of them can lead to pathological situations. For example, the theory of small categories with small limits is not categorically interesting, as they are merely preordered sets. However, a careful bookkeeping of size has a substantial payoff: It gives rise to the theory of *locally presentable categories*. These categories contain most large categories of interest, while enjoying good categorical properties (bicomplete, adjoint functor theorems hold, stable under exponential, etc.).

Presentable ∞ -categories are an ∞ -categorical generalization of locally presentable categories. As in the 1-categorical setting, they include most large ∞ -categories of interest (spaces, spectra, derived ∞ -categories, sheaves, etc.). At the same time, they retain excellent formal properties. For these reasons, they are now understood as a natural setting in which most homotopy-invariant constructions take place.

Historically, however, the presentable ∞ -categories did not arise directly as a response to size issues in homotopy theory. Instead, they were predated by *combinatorial model categories*, which are model categories satisfying some size conditions. The advantage of combinatorial model categories is that many homotopy-theoretic constructions can be performed within an *ordinary* category, where many arguments and constructions are significantly more tractable than in ∞ -categories. Because of this, they also remain a popular setting for homotopy-theoretic arguments.

This parallel development raises the question of how presentable ∞ -categories and combinatorial model categories are related. In this direction, Dugger and Lurie showed that an ∞ -category is presentable if and only if it underlies a combinatorial model category [Dug01a, Lur09]. More recently, Pavlov [Pav25] upgraded this result by showing that the homotopy theory of combinatorial model categories is equivalent to that of presentable ∞ -categories. Informally, this establishes combinatorial model categories provide as concrete models for presentable ∞ -categories.

In many cases, both ∞ -categories and model categories come with symmetric monoidal structures. This naturally leads to the notions of *presentably symmetric monoidal ∞ -categories* and *combinatorial symmetric monoidal model categories*. Pavlov conjectured that the homotopy theories of these two are likewise equivalent [Pav25, Conjecture 1.9]. Establishing such an equivalence is important not only for conceptual completeness, but also because symmetric monoidal structures play a central role in modern homotopy theory, including operad theory, enrichment, and multiplicative constructions.

The main result of this paper confirms Pavlov's conjecture. More precisely, the main result of this paper proves the following:

Theorem A (Theorem 2.4). *The functor*

$$L: \text{CombSMMC}[\text{Quillen.eq}^{-1}] \rightarrow \text{PrSM}$$

is an equivalence, where:

- *CombSMMC* denotes the category of combinatorial symmetric monoidal model categories and left Quillen symmetric monoidal functors;
- *Quillen.eq* denotes the class of Quillen equivalences;
- *PrSM* denotes the ∞ -category of presentably symmetric monoidal ∞ -categories; and
- *L* carries combinatorial symmetric monoidal model categories to their underlying symmetric monoidal ∞ -category.

Remark 0.1. In an earlier work, Nikolaus–Sagave showed that *L* is essentially surjective [NS17].

Remark 0.2. Theorem 2.4 admits many variations. Precise statements will appear in Section 6. In particular, we will prove the following:

- Every tractable (non-symmetric) monoidal semi-model category is monoidally Quillen equivalent to a simplicial one. This gives a partial solution to an open problem by Hovey [Hov, Problem 10] (Remark 6.5).
- Every presentably symmetric monoidal ∞ -category \mathcal{C} admits an essentially unique presentation by a combinatorial symmetric monoidal model category (Remark 6.3)

Beyond its intrinsic theoretical importance, Theorem A provides an effective way to work with presentably symmetric monoidal ∞ -categories. Theorem A allows us to reduce general statements about them to corresponding statements about combinatorial symmetric monoidal model categories, where concrete constructions are often more manageable. In forthcoming work [Arab], we apply this strategy to establish a new equivalence of two models of enriched ∞ -operads.

Our method. Our approach to Theorem A is similar in spirit to that of Pavlov, but extending his argument to the symmetric monoidal setting presents new difficulties. His approach is built upon two inputs. The first is the work of Barwick–Kan [BK12b], which says that ∞ -categories can be modeled by relative ∞ -categories. The second is the work of Dugger and Low on presenting combinatorial model categories [Dug01a, Low16]. To prove Theorem A, we must extend both of these works to the symmetric monoidal setting. We will replace the first by the author’s earlier work on symmetric monoidal relative categories [Araa], and the second by the new model category of symmetric cubical sets (Subsection 5.1).

Organization of the paper. This paper has 6 sections in the main body and 3 sections in the appendix.

- In Section 1, we recall the definition of monoidal model categories and their variations.
- In Section 2, we state the main theorem, and then show that it follows from three separate propositions. The proof of these propositions will be given in the next three sections (Section 3, 4, and 5). The appendix contains some miscellaneous results that will be used in the paper.
- In Section 6, we give variations of the main result of this paper. We

Notation and convention.

- In addition to the ZFC axioms, we will assume the existence of three nested Grothendieck universes whose elements are called small sets, large sets, and

very large sets.¹ All locally presentable categories are assumed to be large but not very large, so that the collection of locally presentable categories themselves form a very large set.

- We assume that model categories are at most large.
- We use the term “regular cardinal” to mean “small regular cardinal.”
- By ∞ -categories, we mean *quasicategories* in the sense of [Joy02, Lur09]. We mostly follow the terminology and notation of [Lur09].
- We let Fin_* denote the category of finite pointed sets $\langle n \rangle = (\{*, 1, \dots, n\}, *)$ for $n \geq 0$ and pointed maps.
- We let $\mathcal{SM}\mathcal{Cat}_\infty$ denote the ∞ -category of small symmetric monoidal ∞ -categories (defined as the localization of the model category described in [Lur17, Variant 2.1.4.13]). We write $\widehat{\mathcal{SM}\mathcal{Cat}_\infty}$ for the ∞ -category of large symmetric monoidal ∞ -categories.
- We write $\mathcal{PSM} \subset \widehat{\mathcal{SM}\mathcal{Cat}_\infty}$ for the subcategory spanned by the presentably symmetric monoidal ∞ -categories and symmetric monoidal functors that preserve small colimits. We define the ∞ -category \mathcal{PMon} of presentably monoidal ∞ -categories similarly.
- We will not notationally distinguish between ordinary categories and their nerves. We will also regard every $(2, 1)$ -category as an ∞ -category by taking its Duskin nerve. (Recall that this converts $(2, 1)$ -categories into ∞ -categories, and it is functorial in strictly unitary pseudofunctors [Lur25, Tag 00AU].)
- If \mathbf{M} is a model category, we write $\mathbf{M}_{\text{cof}} \subset \mathbf{M}$ for the full subcategory of cofibrant objects. We also write \mathbf{M}_∞ for the underlying ∞ -category of \mathbf{M} (i.e., localization at weak equivalences).
- Given a pair of symmetric ∞ -monoidal categories \mathcal{C}, \mathcal{D} , we let $\text{Fun}^\otimes(\mathcal{C}, \mathcal{D})$ denote the ∞ -category of symmetric monoidal functors $\mathcal{C} \rightarrow \mathcal{D}$. We use similar notation for symmetric monoidal categories.²

1. MONOIDAL MODEL CATEGORIES AND THEIR VARIATIONS

The use of the term “monoidal model category” and its variations is not entirely standardized in the literature. The goal of this section is to record the the precise definitions we will use.

1.1. Plain case. We start with the definition of monoidal model categories. There are several competing definitions in the literature: Most definitions in the literature require the pushout-product axiom, but they often differ in their requirement on how “flat” the unit object should be. In some literature (e.g., [Lur17]), the unit object is required to be cofibrant. This assumption is convenient for theoretical purposes but ends up excluding many interesting examples, such as [Examples]. At the other extreme (e.g., [PS18]), no condition on the unit object is assumed. In this paper, we go for a middle ground by adopting an axiom satisfied by most monoidal model categories we encounter in practice.

Definition 1.1. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be model categories. A functor $F: \mathbf{A} \times \mathbf{B} \rightarrow \mathbf{C}$ is called a **left Quillen bifunctor** if it satisfies the following pair of conditions:

¹Although Pavlov did not make this assumption in his paper [Pav25], it seems extremely inconvenient to drop this assumption due to the nature of the statements we will prove. For example, the statement of the main theorem itself needs to be adjusted without this.

²When \mathcal{C} and \mathcal{D} are symmetric monoidal categories regarded as symmetric monoidal ∞ -categories, then $\text{Fun}^\otimes(\mathcal{C}, \mathcal{D})$ has potentially two meanings. This is not confusing because the two categories are naturally equivalent, although we interpret $\text{Fun}^\otimes(\mathcal{C}, \mathcal{D})$ as the category of ordinary symmetric monoidal functors unless stated otherwise.

- (1) The functor F preserves small colimits in each variable.
- (2) For every pair of cofibrations $f: A \rightarrow A'$ in \mathbf{A} and $g: B \rightarrow B'$ in \mathbf{B} , the map

$$F(A, B') \amalg_{F(A, B)} F(A', B) \rightarrow F(A, B)$$

is a cofibration, and moreover it is a weak equivalence if f or g is one.

Definition 1.2.

- A **monoidal model category** is a biclosed monoidal category $(\mathbf{M}, \otimes, \mathbf{1})$ equipped with a model structure, satisfying the following axioms:
 - (1) (**Pushout-product axiom**) The functor $\otimes: \mathbf{M} \times \mathbf{M} \rightarrow \mathbf{M}$ is a left Quillen bifunctor (Definition 1.1).
 - (2) (**Muro's unit axiom** [Mur15]) There is a weak equivalence $q: \tilde{\mathbf{1}} \rightarrow \mathbf{1}$, with $\tilde{\mathbf{1}}$ cofibrant, such that for every object $X \in \mathbf{M}$, the maps $X \otimes q$ and $q \otimes X$ are weak equivalences.
- We say that a monoidal model category is **combinatorial** if its underlying model category is combinatorial. If further it admits generating sets of cofibrations and trivial cofibrations whose elements have cofibrant domain, it is called **tractable**.
- If \mathbf{M} and \mathbf{N} are monoidal model categories, then a **monoidal left Quillen functor** is a monoidal functor³ $\mathbf{M} \rightarrow \mathbf{N}$ whose underlying functor of model categories is left Quillen. We will write $\text{Fun}^{\otimes, LQ}(\mathbf{M}, \mathbf{N})$ for the category of monoidal left Quillen functors $\mathbf{M} \rightarrow \mathbf{N}$ and monoidal natural transformations between them.
- We will write **CombMMC** for the (very large) category of combinatorial monoidal model categories and monoidal left Quillen functors, and write **CombMMC**¹ \subset **CombMMC** for the full subcategory spanned by the combinatorial monoidal model categories with a cofibrant unit.
- **Symmetric monoidal model categories** and **symmetric monoidal left Quillen functors** are defined similarly. (We will use the notation $\text{Fun}^{\otimes, LQ}(-, -)$ in the symmetric monoidal case, too. This is abusive but is rarely confusing.) The categories **CombSMMC** and **CombSMMC**¹ are also defined similarly, using combinatorial symmetric monoidal model categories.
- We define categories **TractMMC**, **TractSMMC**¹, etc, similarly, using tractable symmetric or non-symmetric monoidal model categories.

The reason why we adopt Muro's unit axiom is that it allows us to turn every monoidal model category into one with a cofibrant unit. More precisely, we have the following theorem.

Theorem 1.3. [Mur15, Theorem 1, Proposition 12] *The inclusion $\text{CombMMC}^1 \subset \text{CombMMC}$ admits a left adjoint. Moreover, the unit of this adjunction preserves and reflects weak equivalences and induces an isomorphism of underlying monoidal categories. A similar claim holds for combinatorial symmetric monoidal model categories.*

The categories **CombMMC** can be upgraded to a $(2, 1)$ -category **CombMMC**_(2,1), whose mapping groupoid are given by the maximal subgroupoid $\text{Fun}^{\otimes, LQ}(\mathbf{M}, \mathbf{N})^{\cong} \subset \text{Fun}^{\otimes, LQ}(\mathbf{M}, \mathbf{N})$. We will use similar notations for the categories introduced in Definition 1.2. These ∞ -categories satisfy the following universal property:

³By a monoidal functor, we mean a strong monoidal functor in the sense of [ML98, Chapter XI].

Proposition 1.4. *The functors*

$$\begin{aligned} \text{CombMMC} &\rightarrow \text{CombMMC}_{(2,1)}, \\ \text{CombMMC}^1 &\rightarrow \text{CombMMC}_{(2,1)}^1, \\ \text{CombSMMC} &\rightarrow \text{CombSMMC}_{(2,1)}, \\ \text{CombSMMC}^1 &\rightarrow \text{CombSMMC}_{(2,1)}^1 \end{aligned}$$

are ∞ -categorical localizations at the left Quillen functors whose underlying functors are equivalences of categories.

Proof. We will prove the assertion for the first functor; the remaining assertions can be proved in a similar way. According to (the dual of) [ACK25, Corollary 3.16], it suffices to show that $\text{CombMMC}_{(2,1)}$ admits weak cotensor by [1]. That is, it suffices to show that for each $\mathbf{N} \in \text{CombMMC}_{(2,1)}$, there is an object $[1] \pitchfork \mathbf{N}$ admitting an isomorphism of groupoids

$$\text{Fun}^{\otimes, LQ}(\mathbf{M}, [1] \pitchfork \mathbf{N})^{\cong} \cong \text{Fun}([1], \text{Fun}^{\otimes, LQ}(\mathbf{M}, \mathbf{N})^{\cong})$$

natural in $\mathbf{M} \in \text{CombMMC}$. For this, we consider the groupoid J with two objects 0, 1, and with exactly one morphism between any pair of objects. We can make \mathbf{N}^J into a monoidal model category by using the degreewise tensor product and the model structure coming from the equivalence of categories $\mathbf{N} \xrightarrow{\sim} \mathbf{N}^J$. Then \mathbf{N}^J is a weak cotensor of \mathbf{N} by [1], and we are done. \square

1.2. Enriched case. We next turn to the definition of enriched monoidal model categories.

Definition 1.5. Let \mathbf{V} be a combinatorial symmetric monoidal model category.

- A **\mathbf{V} -monoidal model category** is a tensored and cotensored \mathbf{V} -monoidal category \mathbf{M} (in the sense of [Day70]) which is biclosed (i.e., for each $X \in \mathbf{M}$, the \mathbf{V} -functors $X \otimes -$ and $- \otimes X$ admit right adjoints), equipped with a model structure on its underlying category, such that the tensor bifunctor

$$\mathbf{V} \times \mathbf{M} \rightarrow \mathbf{M}$$

is a left Quillen bifunctor.

- We say that a \mathbf{V} -monoidal model category is **combinatorial** if its underlying model category is combinatorial.
- If \mathbf{M} and \mathbf{N} are \mathbf{V} -monoidal model categories, we write $\text{Fun}_{\mathbf{V}}^{\otimes, LQ}(\mathbf{M}, \mathbf{N})$ for the category of \mathbf{V} -monoidal functors $\mathbf{M} \rightarrow \mathbf{N}$ that are \mathbf{V} -cocontinuous and whose underlying functors are left Quillen.
- We write $\text{CombMMC}_{\mathbf{V}, (2,1)}$ for the $(2, 1)$ -category whose objects are the combinatorial \mathbf{V} -monoidal model categories, with hom-groupoids given by the maximal subgroupoids of $\text{Fun}_{\mathbf{V}}^{\otimes, LQ}(-, -)$.
- We write $\text{CombMMC}_{\mathbf{V}}$ for the underlying category of $\text{CombMMC}_{\mathbf{V}, (2,1)}$.

We define \mathbf{V} -symmetric monoidal model categories similarly, and define a $(2, 1)$ -category $\text{CombSMMC}_{\mathbf{V}, (2,1)}$ and an ordinary category $\text{CombSMMC}_{\mathbf{V}}$ similarly. As in Definition 1.2, we also consider full subcategories of these categories such as $\text{TractMMC}_{\mathbf{V}}$ and $\text{TractSMMC}_{\mathbf{V}}^1$.

In ordinary algebra, a commutative algebra A over a commutative ring k can be defined as a commutative ring equipped with a ring homomorphism $k \rightarrow A$. Extending an analogy of this to the model-categorical setting, we arrive at the notion of *symmetric algebras*:

Definition 1.6. Let \mathbf{V} be a combinatorial symmetric monoidal model category.

- A **symmetric \mathbf{V} -algebra** is a monoidal model category \mathbf{M} equipped with a monoidal left Quillen functor $\mathbf{V} \rightarrow \mathbf{M}$.
- We say that a symmetric \mathbf{V} -algebra is **combinatorial** if its underlying model category is combinatorial.
- We write $\text{CombSym}\mathbf{V}\text{-Alg}_{(2,1)} = \int^{\mathbf{M} \in \text{CombSMMC}_{(2,1)}} \text{Fun}^{\otimes, LQ}(\mathbf{V}, \mathbf{M})$ for the $(2,1)$ -category of combinatorial symmetric \mathbf{V} -algebras, where \int denotes the 2-Grothendieck construction [Str80, 1.10]. Explicitly:
 - Objects are combinatorial symmetric \mathbf{V} -algebras.
 - A 1-morphism $(F: \mathbf{V} \rightarrow \mathbf{M}) \rightarrow (G: \mathbf{V} \rightarrow \mathbf{N})$ is a pair (H, α) , where $H: \mathbf{M} \rightarrow \mathbf{N}$ is a symmetric monoidal functor and $\alpha: HF \Rightarrow G$ is a symmetric monoidal natural transformation.
 - A 2-morphism $(H, \alpha) \Rightarrow (H', \alpha'): (F: \mathbf{V} \rightarrow \mathbf{M}) \rightarrow (G: \mathbf{V} \rightarrow \mathbf{N})$ is a symmetric monoidal natural isomorphism $\beta: H \xrightarrow{\cong} H'$ satisfying $\alpha' \circ \beta F = \alpha$:

$$\begin{array}{ccc}
\begin{array}{c} \mathbf{V} \\ \swarrow F \quad \searrow G \\ \mathbf{M} \xrightarrow{H} \mathbf{N} \end{array} & = & \begin{array}{c} \mathbf{V} \\ \swarrow F \quad \searrow G \\ \mathbf{M} \xrightarrow{H} \mathbf{N} \\ \text{with} \\ \text{an} \\ \text{isomorphism} \\ \beta: H \xrightarrow{\cong} H' \\ \text{such that} \\ \alpha' \circ \beta F = \alpha \end{array}
\end{array}$$

- We write $\text{CombSym}\mathbf{V}\text{-Alg}$ for the underlying category of $\text{CombSym}\mathbf{V}\text{-Alg}_{(2,1)}$.

Remark 1.7. There is a related notion of *central algebras* [Hov99, Definition 4.1.10], but they are not amenable to the techniques we use in this paper.

As in Proposition 1.4, we have:

Proposition 1.8. *Let \mathbf{V} be a combinatorial symmetric monoidal model category. The functors*

$$\begin{aligned}
\text{CombMMC}_{\mathbf{V}} &\rightarrow \text{CombMMC}_{\mathbf{V},(2,1)}, \\
\text{CombMMC}_{\mathbf{V}}^1 &\rightarrow \text{CombMMC}_{\mathbf{V},(2,1)}^1, \\
\text{CombSMMC}_{\mathbf{V}} &\rightarrow \text{CombSMMC}_{\mathbf{V},(2,1)}, \\
\text{CombSMMC}_{\mathbf{V}}^1 &\rightarrow \text{CombSMMC}_{\mathbf{V},(2,1)}^1, \\
\text{CombSym}\mathbf{V}\text{-Alg} &\rightarrow \text{CombSym}\mathbf{V}\text{-Alg}_{(2,1)}
\end{aligned}$$

are all localizations.

1.3. Semi-model-categorical case. (Left) **Semi-model categories** are a weakening of Quillen model categories in which part of the axioms of lifting and factorization are only required to hold for cofibrations with *cofibrant* source [BW24, Definition 2.1]. While this leads to a less attractive set of axioms, semi-model categories are practically indistinguishable from model categories. The main advantages of semi-model categories over ordinary model categories is that they are much easier to construct than a full model structure. In particular, tractable semi-model categories are fully compatible with (left) Bousfield localizations [BW24, Theorem A]. For these reasons, semi-model categories have become increasingly popular, and we have no reason not to include them in this paper.

Definition 1.9. A semi-model category is said to be **tractable** if it is locally presentable as a category and admits generating sets of cofibrations of cofibrant objects and trivial cofibrations of cofibrant objects.

Definition 1.10. A **left Quillen functor** $M \rightarrow N$ of tractable semi-model categories is a left adjoint that preserves cofibrations of cofibrant objects and trivial cofibrations of cofibrant objects. (Equivalently, its right adjoint preserves fibrations and trivial fibrations.)

A **left Quillen bifunctor** $F: M \times N \rightarrow P$ of tractable semi-model categories is a functor that preserves small colimits in each variable, and which satisfies the pushout-product axiom for cofibrations and trivial cofibrations of cofibrant objects and trivial cofibrations of cofibrant objects.

Definition 1.11. The definition of monoidal model categories carries over to the semi-model categorical case. We define a category $\text{TractMMC}_{\text{semi}}$ of tractable monoidal semi-model categories, and define categories such as $\text{TractMMC}_{\text{semi}}^1$, $\text{TractSMMC}_{\text{semi}}$, and $\text{TractSMMC}_{\text{semi}}^1$ similarly, as in Definition 1.2. Also, for a combinatorial symmetric monoidal semi-model category \mathbf{V} , there is a notion of \mathbf{V} -monoidal model categories and \mathbf{V} -symmetric monoidal model categories. The categories $\text{TractMMC}_{\mathbf{V},\text{semi}}^1$ and $\text{TractSMMC}_{\mathbf{V},\text{semi}}$, etc, are defined exactly as in Definition 1.5.

Remark 1.12. Theorem 1.3 and Proposition 1.4 remains valid for semi-model categories, with the same proof.

2. MAIN RESULT

In this section, we state the main result of this paper and explain how we will prove it.

To state our main result, we need a bit of terminology and notation.

Definition 2.1. [Araa, Definition 0.1] A **symmetric monoidal relative category** is a symmetric monoidal category \mathbf{C} equipped with a subcategory $\mathbf{W} \subset \mathbf{C}$ of **weak equivalences**, which contains all isomorphisms and is stable under tensor products.

We write SMRelCat for the category of small symmetric monoidal relative categories and symmetric monoidal relative functors that preserve weak equivalences. We define the category $\widehat{\text{SMRelCat}}$ of *large* symmetric monoidal relative categories similarly.

Example 2.2. Let M be a symmetric monoidal model category. Weak equivalences of M are generally not stable under tensor products, so M is generally *not* a symmetric monoidal relative category. Nonetheless, its full subcategory $M^\flat \subset M$ of cofibrant objects and the objects isomorphic to the unit objects is a symmetric monoidal relative category.

For the next definition, we recall that localization of symmetric monoidal relative categories determines a functor $L: \text{SMRelCat} \rightarrow \widehat{\text{SMCat}}_\infty$ [Araa, Notation 2.1].

Definition 2.3. We define a functor

$$(-)_\infty: \text{CombSMMC} \rightarrow \text{PrSM}$$

as the codomain restriction of the composite

$$\text{CombSMMC} \xrightarrow{(-)^\flat} \widehat{\text{SMRelCat}} \xrightarrow{L} \widehat{\text{SMCat}}_\infty.$$

(Note that this is well-defined by [Lur17, Proposition 1.3.4.22 and Corollary 1.3.4.26].) If M is a combinatorial symmetric monoidal model category, we refer to M_∞ as its **underlying symmetric monoidal ∞ -category**.

We now arrive at the statement of the main result.

Theorem 2.4. *The functor $(-)_\infty$ of Definition 2.3 is a localization at Quillen equivalences, i.e., it induces a categorical equivalence*

$$\text{CombSMMC}[\text{Quill.eq}^{-1}] \xrightarrow{\sim} \text{PrSM}.$$

We will prove Theorem 2.4 as follows: Fix a left proper tractable symmetric monoidal model category \mathbf{S} with a cofibrant unit, whose underlying symmetric monoidal ∞ -category is equivalent to the cartesian monoidal ∞ -category \mathcal{S} of ∞ -groupoids.⁴ (Examples include the cartesian model category of simplicial sets with the Kan–Quillen model structure.) We also define a category $\widehat{\text{SMRelCat}}_{\text{PrSM}}$ by the pullback

$$\begin{array}{ccc} \widehat{\text{SMRelCat}}_{\text{PrSM}} & \longrightarrow & \widehat{\text{SMRelCat}} \\ \downarrow & \lrcorner & \downarrow L \\ \text{PrSM} & \longrightarrow & \widehat{\text{SMCat}}_\infty. \end{array}$$

We then contemplate the following commutative diagram

$$\begin{array}{ccccc} \text{CombSymS-Alg} & \xrightarrow{U} & \text{CombSMMC}^1 & \xleftarrow{\iota} & \text{CombSMMC} \\ (-)_b \downarrow & & & & \downarrow (-)_\infty \\ \widehat{\text{SMRelCat}}_{\text{PrSM}} & \xrightarrow{L} & & & \widehat{\text{PrSMCat}}_\infty, \end{array}$$

where U is the forgetful functor and ι is the inclusion. We will then prove that:

Proposition 2.5. *The functor $L: \widehat{\text{SMRelCat}}_{\text{PrSM}} \rightarrow \widehat{\text{PrSMCat}}_\infty$ induces a categorical equivalence*

$$\widehat{\text{SMRelCat}}_{\text{Pr}}[\text{loc.eq}^{-1}] \rightarrow \widehat{\text{PrSMCat}}_\infty,$$

where loc.eq denotes the subcategory of **local equivalences**, i.e., morphisms inverted by L .

Proposition 2.6. *The functor $(-)_b$ induces a categorical equivalence*

$$\text{CombSymS-Alg}[\text{Quill.eq}^{-1}] \xrightarrow{\sim} \widehat{\text{SMRelCat}}_{\text{PrSM}}[\text{loc.eq}^{-1}].$$

Proposition 2.7. *The functor U induces a categorical equivalence*

$$\text{CombSymS-Alg}[\text{Quill.eq}^{-1}] \xrightarrow{\sim} \text{CombSMMC}^1[\text{Quill.eq}^{-1}].$$

The proof of these propositions will be given in the next three sections (Section 3, 4, and 5). Since we know that ι induces a categorical equivalence upon localizing at Quillen equivalences (Theorem 1.3), these proposition will prove Theorem 2.4.

3. PROOF OF PROPOSITION 2.5

We first prove Proposition 2.5, which asserts that the functor

$$\widehat{\text{SMRelCat}}_{\text{PrSM}}[\text{loc.eq}^{-1}] \rightarrow \text{PrSM}$$

is an equivalence. In fact, we will prove the following more general assertion:

Lemma 3.1. *For every conservative functor $\mathcal{X} \rightarrow \widehat{\text{SMCat}}_\infty$ of ∞ -categories, the functor*

$$\text{SMRelCat}_{\mathcal{X}} = \mathcal{X} \times_{\widehat{\text{SMCat}}_\infty} \widehat{\text{SMRelCat}} \rightarrow \mathcal{X}$$

is a localization.

⁴Since \mathcal{S} is the initial presentably symmetric monoidal ∞ -category [Lur17, Corollary 3.2.1.9, Example 4.8.1.20], such an equivalence is unique up to a contractible space of choices if it exists.

Replacing $\mathcal{SM}\mathcal{C}\mathbf{at}_\infty$ by $\widehat{\mathcal{SM}\mathcal{C}\mathbf{at}}_\infty$ and then substituting $\mathcal{P}\mathcal{R}\mathcal{SM}$ for \mathcal{X} , we obtain Proposition 2.5.

For the proof of Lemma 3.1, we need some terminology and notation.

- A **relative ∞ -category** is an ∞ -category equipped with a subcategory $\mathcal{W} \subset \mathcal{C}$ containing all equivalences. Morphisms in \mathcal{W} are typically called **weak equivalences**. When \mathcal{C} is (the nerve of) an ordinary category, we call such pairs **relative categories**. A relative functor is a functor. This should not be confused with the definition of Barwick–Kan [BK12b], which they only require that \mathcal{W} contains all objects.
- If \mathcal{C} and \mathcal{D} are relative ∞ -categories, a **relative functor** $f: \mathcal{C} \rightarrow \mathcal{D}$ is a functor between the underlying ∞ -categories that preserve weak equivalences. We say that such an f is a **homotopy equivalence** if there is a relative functor $g: \mathcal{D} \rightarrow \mathcal{C}$ and zig-zags of natural weak equivalences connecting $f \circ g$ and $g \circ f$ to the identity functors.
- We write \mathbf{RelCat} for the category of small relative categories and functors preserving weak equivalences. For disambiguation, we will write $\mathbf{RelCat}_{\mathbf{BK}}$ for the slightly larger category of Barwick–Kan's relative categories that are small.
- We call a morphism $f: \mathcal{C} \rightarrow \mathcal{D}$ of $\mathbf{RelCat}_{\mathbf{BK}}$ a **local equivalences** if it induces an equivalence between the (∞ -categorical) localizations.
- Recall that a symmetric monoidal category is called a **permutative category** if its underlying monoidal category is strict (i.e., its associators and the unitors are the identity maps). We write $\mathbf{PermRelCat} \subset \mathbf{SMRelCat}$ for the subcategory of objects whose underlying symmetric monoidal categories are permutative categories, and morphisms whose underlying symmetric monoidal functors are strict.
- We will regard the categories \mathbf{RelCat} , $\mathbf{RelCat}_{\mathbf{BK}}$, $\mathbf{PermRelCat}$, $\mathbf{SMRelCat}$ as relative categories whose weak equivalences are the local equivalences.
- We write \mathbf{Fin}_* for the category of finite pointed sets and pointed maps. We write $\mathbf{Fun}^{\mathbf{Seg}}(\mathbf{Fin}_*, \mathbf{RelCat}) \subset \mathbf{Fun}^{\mathbf{Seg}}(\mathbf{Fin}_*, \mathbf{RelCat})$ for the full subcategory spanned by the functors $F: \mathbf{Fin}_* \rightarrow \mathbf{RelCat}$ satisfying the following **Segal condition**: For every $n \geq 0$, the map $\rho_i: \langle n \rangle \rightarrow \langle 1 \rangle$ defined by

$$\rho_i(j) = \begin{cases} * & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases}$$

induces a local equivalence

$$F\langle n \rangle \xrightarrow{\sim} \prod_{1 \leq i \leq n} F\langle 1 \rangle.$$

We define $\mathbf{Fun}^{\mathbf{Seg}}(\mathbf{Fin}_*, \mathbf{RelCat}_{\mathbf{BK}}) \subset \mathbf{Fun}^{\mathbf{Seg}}(\mathbf{Fin}_*, \mathbf{RelCat}_{\mathbf{BK}})$ similarly.

Proof of Lemma 3.1. Since symmetric categories are functorially equivalent to permutative categories [May78, Proposition 4.2], using Proposition B.3, we may replace $\mathbf{SMRelCat}$ by $\mathbf{PermRelCat}$.

According to [Araa, Theorem 1.1], there is a homotopy equivalence $\mathbf{Fact}: \mathbf{PermRelCat} \rightarrow \mathbf{Fun}^{\mathbf{Seg}}(\mathbf{Fin}_*, \mathbf{RelCat})$ of relative categories. The proof of loc. cit. further shows that the localization of $\mathbf{Fun}^{\mathbf{Seg}}(\mathbf{Fin}_*, \mathbf{RelCat})$ is equivalent to $\mathcal{SM}\mathcal{C}\mathbf{at}_\infty$, and the diagram

$$\begin{array}{ccc} \mathbf{PermRelCat} & \xrightarrow{\mathbf{Fact}} & \mathbf{Fun}^{\mathbf{Seg}}(\mathbf{Fin}_*, \mathbf{RelCat}) \\ \downarrow & & \downarrow \text{localization} \\ \mathbf{SMRelCat} & \xrightarrow{L} & \mathcal{SM}\mathcal{C}\mathbf{at}_\infty. \end{array}$$

commutes up to natural equivalence. Thus, by Proposition B.3, it will suffice to show that the functor

$$\mathrm{Fun}^{\mathrm{Seg}}(\mathrm{Fin}_*, \mathrm{RelCat})_{\mathcal{X}} \rightarrow \mathcal{X}$$

is a localization. Since the inclusion $\mathrm{RelCat} \hookrightarrow \mathrm{RelCat}_{\mathrm{BK}}$ is a homotopy equivalence of relative categories, we are free to replace RelCat by $\mathrm{RelCat}_{\mathrm{BK}}$.

We now recall from [BK12b, Theorem 6.1] and [BK12a, Theorem 1.8] that $\mathrm{RelCat}_{\mathrm{BK}}$ and its local equivalences is part of a combinatorial model structure. Therefore, Theorem B.1 and [Hir03, Theorem 11.6.1] show that the localization

$$\mathrm{Fun}(\mathrm{Fin}_*, \mathrm{RelCat}_{\mathrm{BK}}) \rightarrow \mathrm{Fun}(\mathrm{Fin}_*, \mathrm{RelCat}_{\mathrm{BK}})[\mathrm{weq}^{-1}]$$

is stable under pullback. In particular, the functor

$$\mathrm{Fun}^{\mathrm{Seg}}(\mathrm{Fin}_*, \mathrm{RelCat}_{\mathrm{BK}}) \rightarrow \mathcal{X}$$

is a localization, which was to be proved. \square

4. PROOF OF PROPOSITION 2.6

Notation 4.1. Throughout this section, we fix a left proper tractable symmetric monoidal model category \mathbf{S} with a cofibrant unit, whose underlying symmetric monoidal ∞ -category is equivalent to the cartesian monoidal ∞ -category \mathcal{S} of ∞ -groupoids.

In this section, we give a proof of Proposition 2.6, which asserts that the functor

$$\mathrm{CombSymS-Alg}[\mathrm{Quill.eq}^{-1}] \xrightarrow{\sim} \widehat{\mathrm{SMRelCat}_{\mathrm{PrSM}}}[\mathrm{loc.eq}^{-1}]$$

is an equivalence. This will need two ingredients: A relative-categorical version of the multiplicative Gabriel–Ulmer duality, and some formal cardinality argument of model categories. We will tackle each of these in the next two subsections, and then prove Proposition 2.6 at the end of this subsection.

4.1. Relative-categorical multiplicative Gabriel–Ulmer duality. In this subsection, we state and prove a relative-categorical multiplicative Gabriel–Ulmer duality (Proposition 4.6). We refer the reader to Section A for a review of Gabriel–Ulmer duality.

To state the main result, we need a bit of preliminaries.

Notation 4.2. Let κ be a regular cardinal. We write $\mathrm{SMCat}_\infty(\kappa)$ for the subcategory of SMCat_∞ spanned by the symmetric monoidal ∞ -categories that compatible with κ -small colimits (i.e., those ∞ -categories that have κ -small colimits and whose tensor product preserves those colimits in each variable).

We write $\mathrm{SMCat}_\infty^{\mathrm{idem}}(\kappa) \subset \mathrm{SMCat}_\infty(\kappa)$ for the full subcategory spanned by the objects that are idempotent complete.

Construction 4.3. Let κ be a regular cardinal. For each $\mathcal{C} \in \mathrm{SMRelCat}_{\mathrm{SMCat}_\infty^{\mathrm{idem}}(\kappa)}$, we define an \mathbf{S} -algebra $\mathrm{Ind}_\kappa^{\mathbf{S}}(L(\mathcal{C}))$ as follows:

- Its underlying category is $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{S})$.
- The symmetric monoidal structure comes from Day’s convolution product.
- The symmetric monoidal functor $\mathbf{S} \rightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{S})$ is given by $X \mapsto \mathcal{C}(-, \mathbf{1}) \cdot X$, where $\mathbf{1} \in \mathcal{C}$ denote the monoidal unit and the dot “.” denotes copowering by set.
- The model structure is the left Bousfield localization of the projective model structure on $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{S})$ whose weak equivalences are the maps inverted by the composite

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{S}) \rightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \rightarrow \mathrm{Fun}(L(\mathcal{C})^{\mathrm{op}}, \mathcal{S}) \rightarrow \mathrm{Ind}_\kappa(L(\mathcal{C})).$$

Here, the first functor is the postcomposition of the localization $\mathbf{S} \rightarrow \mathcal{S}$, the second functor is the left adjoint Kan extension along the localization functor $\mathcal{C}^{\text{op}} \rightarrow L(\mathcal{C})^{\text{op}}$, and the third functor is the left adjoint to the inclusion $\text{Ind}_{\kappa}(L(\mathcal{C})) \hookrightarrow \text{Fun}(L(\mathcal{C}), \mathcal{S})$.

We sometimes use the notation $\text{Ind}_{\kappa}^{\mathbf{S}}(L(\mathcal{C}))$ for essentially small symmetric monoidal relative categories whose symmetric monoidal localization is idempotent complete and is compatible with κ -small colimits.

Lemma 4.4. *Construction 4.3 is well-defined. More precisely, in the situation of Construction 4.3, the following holds:*

- (1) *The model category $\text{Ind}_{\kappa}^{\mathbf{S}}(L(\mathcal{C}))$ exists.*
- (2) *The model category $\text{Ind}_{\kappa}^{\mathbf{S}}(L(\mathcal{C}))$ satisfies the pushout-product axiom, and the monoidal unit is cofibrant.*
- (3) *The symmetric monoidal functor $\mathbf{S} \rightarrow \text{Ind}_{\kappa}^{\mathbf{S}}(L(\mathcal{C}))$ is left Quillen.*

Proof. Part (1) is a consequence of Proposition B.4.

Next, we prove (2). Choose generating sets I, J of cofibrations and trivial cofibrations of \mathbf{S} . According to [Hir03, Theorem 11.6.1], the sets $I_{\mathcal{C}} = \{\mathcal{C}(-, C) \cdot i\}_{i \in I, C \in \mathcal{C}}$ and $J_{\mathcal{C}} = \{\mathcal{C}(-, C) \cdot j\}_{j \in J, C \in \mathcal{C}}$ generate cofibrations and trivial cofibrations of the projective model structure on $\text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{S})$. It follows immediately that:

- The monoidal unit of $\text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{S})$ is cofibrant (because \mathbf{S} has a cofibrant unit).
- The Day convolution product satisfies the pushout-product axiom for the projective model structure.
- The projective model structure is tractable.

Therefore, to prove (2), it will suffice to prove the following: For every cofibrant object $X \in \text{Ind}_{\kappa}^{\mathbf{S}}(L(\mathcal{C}))$, the functor

$$X \otimes -: \text{Ind}_{\kappa}^{\mathbf{S}}(L(\mathcal{C}))_{\text{cof}} \rightarrow \text{Ind}_{\kappa}^{\mathbf{S}}(L(\mathcal{C}))_{\text{cof}}$$

preserves weak equivalences. But this is clear, because the composite

$$\text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{S})_{\text{cof}} \rightarrow \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{S}) \rightarrow \text{Fun}(L(\mathcal{C})^{\text{op}}, \mathcal{S}) \rightarrow \text{Ind}_{\kappa}(L(\mathcal{C}))$$

is symmetric monoidal.

Claim (3) follows from the fact that its right adjoint is the evaluation at the unit object, which is evidently right Quillen. The claim follows. \square

Notation 4.5. For each subcategory $\mathcal{X} \subset \text{SMCat}_{\infty}$ containing all equivalences, we will write $\text{SMRelCat}_{\mathcal{X}} = \mathcal{X} \times_{\text{SMCat}_{\infty}} \text{SMRelCat}$, and define $\text{SMRelCat}_{(2,1), \mathcal{X}} \subset \text{SMRelCat}_{(2,1)}$ for the sub $(2, 1)$ -category whose mapping groupoids are the components corresponding to the morphisms in $\text{SMRelCat}_{\mathcal{X}}$.

We can now state the main result of this subsection.

Proposition 4.6. *The strictly unitary pseudofunctor*

$$\begin{aligned} \text{SMRelCat}_{\text{SMCat}_{\infty}^{\text{idem}}(\kappa)} &\rightarrow \widehat{\text{SMRelCat}}_{\kappa\text{-}\text{PrSM}, (2,1)} \\ \mathcal{C} &\mapsto \text{Ind}_{\kappa}^{\mathbf{S}}(L(\mathcal{C})), \end{aligned}$$

induces a categorical equivalence

$$\text{SMRelCat}_{\text{SMCat}_{\infty}^{\text{idem}}(\kappa)}[\text{loc.eq}^{-1}] \xrightarrow{\sim} \widehat{\text{SMRelCat}}_{\kappa\text{-}\text{PrSM}, (2,1)}[\text{loc.eq}^{-1}].$$

We need a lemma for the proof of Proposition 4.6. We will use the following notation: For each subcategory $\mathcal{X} \subset \text{SMCat}_{\infty}$ containing all equivalences, we will write $\text{SMRelCat}_{(2,1), \mathcal{X}} \subset \text{SMRelCat}_{(2,1)}$ for the sub $(2, 1)$ -category whose mapping groupoids are the components corresponding to the morphisms in $\text{SMRelCat}_{\mathcal{X}}$. We then have:

Lemma 4.7. *For every subcategory $\mathcal{X} \subset \mathcal{SMCat}_\infty$ containing all equivalences, the inclusion $\mathcal{SMRelCat}_{\mathcal{X}} \hookrightarrow \mathcal{SMRelCat}_{(2,1),\mathcal{X}}$ is a localization.*

Proof. According to [ACK25, Corollary 3.15], it suffices to show that the $(2, 1)$ -category $\mathcal{SMRelCat}_{(2,1),\mathcal{X}}$ admits a weak cotensor by $[1]$. Given an object $\mathcal{C} \in \mathcal{SMRelCat}_{(2,1)}$, a weak cotensor by $[1]$ is given by \mathcal{C}^J , where the tensor product is given by the degreewise tensor product, and the weak equivalences the natural transformations whose components are weak equivalences. \square

We can now prove Proposition 4.6.

Proof of Proposition 4.6. The Yoneda embedding determines a pseudonatural transformation depicted as

$$\begin{array}{ccc} \mathcal{SMRelCat}_{\mathcal{SMCat}_\infty^{\text{idem}}(\kappa)} & \xrightarrow{\text{Ind}_\kappa^{\mathbf{S}}(L(-))_b} & \widehat{\mathcal{SMRelCat}}_{\kappa\text{-}\mathcal{PrSM},(2,1)} \\ & \searrow \quad \quad \quad \swarrow & \\ & \widehat{\mathcal{SMRelCat}}_{(2,1)} & \end{array}$$

Using Lemmas 3.1 and 4.7 to identify the localization of each category at local equivalences, we obtain a functor $I: \mathcal{SMCat}_\infty^{\text{idem}}(\kappa) \rightarrow \kappa\text{-}\mathcal{PrSM}$ and a natural transformation depicted as

$$\begin{array}{ccc} \mathcal{SMCat}_\infty^{\text{idem}}(\kappa) & \xrightarrow{I} & \kappa\text{-}\mathcal{PrSM} \\ & \searrow \quad \quad \quad \swarrow & \\ & \widehat{\mathcal{SMCat}_\infty}(\kappa). & \end{array}$$

By construction, the pair (I, α) satisfies the hypothesis of Corollary A.9. It follows from this corollary that I is an equivalence, and we are done. \square

4.2. Strongly κ -combinatorial model categories. Combinatorial model categories enjoy the following curious property, first articulated by Dugger in [Dug01a] and later expanded by Low [Low16]: Beyond sufficiently large regular cardinals, the distinction between ordinary-categorical notion and ∞ -categorical notions gets blurry. In this subsection, we record several results that embody this principle.

The following definition is essentially due to Low [Low16, Definition 5.1]:

Definition 4.8. Let κ be a regular cardinal. We say that a model category \mathbf{M} is **strongly κ -combinatorial** if there is a regular cardinal $\kappa_0 \triangleleft \kappa$ with the following properties:

- (1) \mathbf{M} is locally κ_0 -presentable.
- (2) \mathbf{M}_κ is closed under finite limits in \mathbf{M} .
- (3) Each hom-set in \mathbf{M}_{κ_0} is κ -small.
- (4) There are κ -small sets of morphisms in \mathbf{M}_{κ_0} that cofibrantly generate the model structure on \mathbf{M} .

The following results summarize the basic properties of strongly κ -combinatorial model categories.

Proposition 4.9. [Low16, Propositions 5.6, 5.12] *For every combinatorial model category \mathbf{M} , there is a regular cardinal κ such that \mathbf{M} is strongly κ -combinatorial. Moreover, in this situation, \mathbf{M}_κ is a model category whose cofibrations, fibrations, and weak equivalences are precisely those of \mathbf{M} of κ -compact objects.*

Proposition 4.10. [Low16, Remark 5.2] *Let $\kappa' \triangleright \kappa$ be regular cardinals. Every strongly κ -combinatorial model category is strongly κ' -combinatorial.*

Proposition 4.11. *Let κ be a regular cardinal, and let \mathbf{M} be a strongly κ -combinatorial model category. Then the ∞ -category $\mathbf{M}_{\kappa,\infty} = (\mathbf{M}_\kappa)_\infty$ admits κ -small colimits, and the functor*

$$\iota: \mathbf{M}_{\kappa,\infty} \rightarrow \mathbf{M}_\infty$$

exhibits \mathbf{M}_∞ as an Ind_κ -completion of $\mathbf{M}_{\kappa,\infty}$.

Proof. The fact that $\mathbf{M}_{\kappa,\infty}$ admits κ -small colimits follows from the argument of [Cis19a, Proposition 7.7.4]. To show that ι induces a categorical equivalence $\text{Ind}_\kappa(\mathbf{M}_{\kappa,\infty}) \xrightarrow{\sim} \mathbf{M}_\infty$, it suffices to prove the following:

- (1) The functor ι is fully faithful.
- (2) The essential image of ι generates \mathbf{M}_∞ under κ -filtered colimits.
- (3) The image of ι lies in $(\mathbf{M}_\infty)_\kappa$.

Claim (1) follows from Proposition 4.9 and [ACK25, Corollary 3.1], which asserts that the classical derived mapping spaces (computed using simplicial resolutions objects) computes the mapping spaces of the localization of model categories.

For (2), we observe that weak equivalences of \mathbf{M} are stable under κ -filtered colimits [RR15, Proposition 4.1]. This implies that κ -filtered colimits in \mathbf{M} are already homotopy colimits, so we only have to show that \mathbf{M}_κ generates \mathbf{M} under κ -filtered colimits. This is clear, because \mathbf{M} is locally κ -presentable.

For (3), suppose we are given a κ -filtered ∞ -category \mathcal{I} and a colimit diagram $\overline{F}: \mathcal{I}^\triangleright \rightarrow \mathbf{M}_\infty$. We must show that, for every object $X \in \mathbf{M}_\kappa$, the diagram $\mathbf{M}_\infty(X, -) \circ \overline{F}: \mathcal{I}^\triangleright \rightarrow \mathcal{S}$ is a colimit diagram. Using [Lur25, Tag 02QA], we may assume that \mathcal{I} is (the nerve of) a poset. In this case, the functor $\text{Fun}(\mathcal{I}^\triangleright, \mathbf{M}) \rightarrow \text{Fun}(\mathcal{I}^\triangleright, \mathbf{M}_\infty)$ is a localization [Cis19a, Theorem 7.9.8], so we may assume that F lifts to a diagram $\overline{G}: \mathcal{I}^\triangleright \rightarrow \mathbf{M}$. Without loss of generality, we may assume that \overline{G} takes values in the full subcategory of fibrant objects. Set $G = \overline{G}|_{\mathcal{I}}$. Since \mathcal{I} is κ -filtered, the map

$$\text{colim}_{\mathcal{I}} G \rightarrow \overline{G}(\infty)$$

is a weak equivalence. Moreover, since fibrations of \mathbf{M} are stable under κ -filtered colimits, the object $\text{colim}_{\mathcal{I}} G \in \mathbf{M}$ is fibrant. Thus, we may assume that \overline{G} is a strict colimit diagram.

Now choose a cosimplicial resolution X^\bullet of X in \mathbf{M}_κ . By [ACK25, Corollary 3.1], the functor $\mathbf{M}(X^\bullet, -): \mathbf{M}_{\text{fib}} \rightarrow \mathbf{sSet}$ descends to the functor $\mathbf{M}_\infty \rightarrow \mathcal{S}$ corepresented by X . Therefore, we are reduced to showing that the diagram $\mathbf{M}(X^\bullet, \overline{G}): \mathcal{I}^\triangleright \rightarrow \mathbf{sSet}$ is a homotopy colimit diagram. This is clear, since each X^\bullet is κ -compact and κ -filtered colimits are homotopy colimits in \mathbf{sSet} . The proof is now complete. \square

Corollary 4.12. *Let κ be a regular cardinal, and let \mathbf{M} be a symmetric \mathbf{S} -algebra whose underlying model category is strongly κ -combinatorial. The inclusion $\mathbf{M}_{\kappa,\text{cof}} \hookrightarrow \mathbf{M}$ induces a left Quillen equivalence of \mathbf{S} -algebras*

$$\begin{aligned} \theta: \text{Ind}_\kappa^{\mathbf{S}}(L(\mathbf{M}_{\kappa,\text{cof}})) &\xrightarrow{\sim} \mathbf{M} \\ X &\mapsto \int^{M \in \mathbf{M}_{\kappa,\text{cof}}} M \otimes X(M). \end{aligned}$$

Proof. We first show that θ induces a left Quillen functor

$$\theta': \text{Fun}\left(\mathbf{M}_{\kappa,\text{cof}}^{\text{op}}, \mathbf{S}\right) \rightarrow \mathbf{M},$$

where $\text{Fun}\left(\mathbf{M}_{\kappa,\text{cof}}^{\text{op}}, \mathbf{S}\right)$ carries the projective model structure. Since \mathbf{M} is strongly κ -combinatorial, it has a generating set I of cofibrations consisting of maps in \mathbf{M}_κ . The maps $\{\mathbf{M}(-, M) \otimes i\}_{M \in \mathbf{M}_{\kappa,\text{cof}}, i \in I}$ generate projective cofibrations of

$\text{Fun}(\mathbf{M}_{\kappa, \text{cof}}^{\text{op}}, \mathbf{S})$ [Hir03, Theorem 11.6.1]. The images of these maps under θ are simply the maps $\{M \otimes i\}_{M \in \mathbf{M}_{\kappa, \text{cof}}, i \in I}$, which are all cofibrations. Hence θ carries projective cofibrations to cofibrations. A similar argument shows that θ carries projective trivial cofibrations to trivial cofibrations. Hence θ' is left Quillen, as claimed.

We now consider the following diagram, which commutes up to natural equivalence:

$$\begin{array}{ccc}
 \mathbf{M}_{\kappa, \text{cof}} & \xrightarrow{\quad} & \mathbf{M}_{\kappa, \infty} \\
 \downarrow & & \downarrow i \\
 \text{Fun}(\mathbf{M}_{\kappa, \text{cof}}^{\text{op}}, \mathbf{S})_{\infty} & \xrightarrow{\theta'_{\infty}} & \mathbf{M}_{\infty} \\
 \downarrow & & \\
 \mathbf{Ind}_{\kappa}^{\mathbf{S}}(L(\mathbf{M}_{\kappa, \text{cof}}))_{\infty}. & &
 \end{array}$$

According to Proposition 4.11, the functor i exhibits \mathbf{M}_{∞} as an Ind_{κ} -completion of $\mathbf{M}_{\kappa, \infty}$. Also, by what we have just shown in the previous paragraph, the functor θ'_{∞} preserves small colimits. It follows that θ'_{∞} can be identified with the functor

$$\text{Fun}(\mathbf{M}_{\kappa, \text{cof}}^{\text{op}}, \mathbf{S}) \rightarrow \text{Ind}_{\kappa}(\mathbf{M}_{\kappa, \text{cof}})$$

described in Construction 4.3. By the definition of weak equivalences of $\mathbf{Ind}_{\kappa}^{\mathbf{S}}(L(\mathbf{M}_{\kappa, \text{cof}}))$, this means that θ preserves weak equivalences of cofibrant objects of $\mathbf{Ind}_{\kappa}^{\mathbf{S}}(L(\mathbf{M}_{\kappa, \text{cof}}))$. Since we already know that θ' is a left Quillen functor, and since $\mathbf{Ind}_{\kappa}^{\mathbf{S}}(L(\mathbf{M}_{\kappa, \text{cof}}))$ is tractable, this is enough to conclude that θ is a left Quillen functor. The resulting functor

$$\theta_{\infty} : \text{Ind}_{\kappa}^{\mathbf{S}}(L(\mathbf{M}_{\kappa, \text{cof}}))_{\infty} \rightarrow \mathbf{M}_{\infty}$$

is an equivalence, because both sides are the Ind_{κ} -completion of $\mathbf{M}_{\kappa, \infty}$. Hence θ is a left Quillen equivalence, and we are done. \square

We conclude this subsection by taking monoidal structures into account.

Definition 4.13. Let κ be a regular cardinal. A **monoidally κ -combinatorial model category** is a monoidal model category \mathbf{M} satisfying the following conditions:

- As a model category, \mathbf{M} is strongly κ -combinatorial.
- κ -compact objects are stable under finite tensor products. (In particular, the unit object is κ -compact.)

We write $\text{CombMMC}(\kappa) \subset \text{CombMMC}$ for the subcategory spanned by the monoidally κ -combinatorial model category and those maps that preserve κ -compact objects.

Lemma 4.14.

- (1) For every pair of regular cardinals $\kappa \triangleleft \lambda$, we have $\text{CombMMC}(\kappa) \subset \text{CombMMC}(\lambda)$.
- (2) Every morphism of CombMMC belongs to $\text{CombMMC}(\kappa)$ for some regular cardinal κ .

Proof. We first show that every combinatorial monoidal model category \mathbf{M} is monoidally κ -combinatorial for some κ , and if this is true, then \mathbf{M} is monoidally λ -combinatorial for every $\lambda \triangleright \kappa$. By Proposition 4.9, there is some regular cardinal κ_0 such that \mathbf{M} is strongly κ_0 -combinatorial as a model category. Find a regular cardinal $\kappa_0 \triangleleft \kappa$ such that, if $X, Y \in \mathbf{M}_{\kappa_0}$, then $X \otimes Y \in \mathbf{M}_{\kappa}$. Since every κ -compact object is a κ -small colimit of κ_0 -compact object [MP89, Theorem 2.3.11], this ensures that κ -compact objects is stable under tensor product. Hence \mathbf{M} is monoidally κ -combinatorial. A similar argument, using Proposition

4.10, shows that every monoidally κ -combinatorial model category is monoidally λ -combinatorial.

We now prove (1) and (2). Part (1) follows from the result in the previous paragraph and [AR94, Remark 2.20]. For (2), let $F: \mathbf{M} \rightarrow \mathbf{N}$ be a morphism of CombMMC . By the result in the previous paragraph and part (1), we can find some κ_0 such that \mathbf{M} and \mathbf{N} are monoidally κ_0 -combinatorial. Choose $\kappa \triangleright \kappa_0$ such that $F(\mathbf{M}_{\kappa_0}) \subset \mathbf{N}_\kappa$. We claim that F belongs to $\text{CombMMC}(\kappa)$.

Since every κ -compact object is a κ -small colimit of κ_0 -compact object [MP89, Theorem 2.3.11], our choice of κ and κ_0 ensure that $F(\mathbf{M}_\kappa) \subset \mathbf{N}_\kappa$. We also know from (1) that \mathbf{M} and \mathbf{N} are monoidally κ -combinatorial from. Thus F belongs to $\text{CombMMC}(\kappa)$, as desired. \square

4.3. Proof of Proposition 2.6. We now arrive at the proof of Proposition 2.6.

Notation 4.15. We write urCard for the (large) poset of (small) uncountable regular cardinals, ordered by \triangleleft . Given a functor $F: \text{urCard} \rightarrow \widehat{\text{Cat}}$, we write $\int^{\text{urCard}} F \rightarrow \text{urCard}$ for the Grothendieck construction of F . Likewise, given a functor $G: \text{urCard}^{\text{op}} \rightarrow \widehat{\text{Cat}}$, its Grothendieck construction will be denoted by $\int_{\text{urCard}} G \rightarrow \text{urCard}$.

Notation 4.16. We let $\text{CombSymS-Alg}^{\text{sk}} \subset \text{CombSymS-Alg}$ denote the full subcategory of \mathbf{S} -algebras \mathbf{M} whose underlying category is skeletal (i.e., isomorphic objects are equal). Note that for such an \mathbf{M} , the full subcategory \mathbf{M}_κ of κ -compact objects is *literally* small for every κ .

We define a full subcategory $\text{CombSymS-Alg}_{(2,1)}^{\text{sk}} \subset \text{CombSymS-Alg}_{(2,1)}$ similarly.

Proof of Proposition 2.6. By Proposition 1.8 and Lemma 4.7, it suffices to show that the functor

$$\text{CombSymS-Alg}_{(2,1)}^{\text{sk}}[\text{Quill.eq}^{-1}] \rightarrow \widehat{\text{SMRelCat}}_{(2,1)}[\text{loc.eq}^{-1}]$$

is an equivalence.

For each uncountable regular cardinal κ , let $\text{CombSymS-Alg}(\kappa)^{\text{sk}}$ denote the fiber product

$$\text{CombSymS-Alg}^{\text{sk}} \times_{\text{CombMMC}} \text{CombMMC}(\kappa).$$

We now consider the following diagram:

$$\begin{array}{ccc} \int^{\text{urCard}} \text{CombSymS-Alg}(-)^{\text{sk}} & \xrightarrow{F} & \int_{\text{urCard}} \widehat{\text{SMRelCat}}_{\text{SMCat}_\infty}(-) \\ U \downarrow & \swarrow \text{Ind}^{\mathbf{S}} & \downarrow (-)_b \circ \text{Ind}^{\mathbf{S}} \\ \text{CombSymS-Alg}_{(2,1)}^{\text{sk}} & \xrightarrow{(-)_b} & \widehat{\text{SMRelCat}}_{\text{PrSM},(2,1)}. \end{array}$$

Here U and F are functors and $\text{Ind}^{\mathbf{S}}$ is a strictly unitary pseudofunctor, defined by the formulas

$$\begin{aligned} F(\kappa, \mathbf{M}) &= (\kappa, \mathbf{M}_{\kappa, \text{cof}}), \\ U(\kappa, \mathbf{M}) &= \mathbf{M}, \\ \text{Ind}^{\mathbf{S}}(\kappa, \mathcal{C}) &= \text{Ind}_\kappa^{\mathbf{S}}(L(\mathcal{C})). \end{aligned}$$

The lower triangle commutes by construction; the upper triangle does not commute on the nose, but Corollary 4.12 gives a pseudonatural Quillen equivalence $\text{Ind}^{\mathbf{S}} \circ F \xrightarrow{\sim} U$. Thus, to prove the claim, it suffices to show that U and $(-)_b \circ \text{Ind}^{\mathbf{S}}$ induce equivalences of ∞ -categories when localized at the maps whose images in PrSM are equivalences.

We start from the claim on U . According to Lemma 4.14, the category CombSymS-Alg is the colimit of $\text{CombSymS-Alg}(\kappa)$ as κ ranges over urCard . It follows from [Lur25, Tag 02UU] that U is already a localization.

Next, for $(-)_b \circ \mathbf{Ind}^S$, we factor it as

$$\begin{aligned} \int_{\text{urCard}} \text{SMRelCat}_{\text{SMCat}_\infty(-)} &\xrightarrow{\Phi} \int_{\text{urCard}} \widehat{\text{SMRelCat}}_{(-)\text{-}\mathcal{P}r\mathcal{S}\mathcal{M},(2,1)} \\ &\xrightarrow{\Psi} \widehat{\text{SMRelCat}}_{\mathcal{P}r\mathcal{S}\mathcal{M}}, \end{aligned}$$

where $\int^{\text{urCard}} \widehat{\text{SMRelCat}}_{(-)\text{-}\mathcal{P}r\mathcal{S}\mathcal{M},(2,1)}$ denotes the cartesian fibration associated with the functor $\kappa \mapsto \widehat{\text{SMRelCat}}_{\kappa\text{-}\mathcal{P}r\mathcal{S}\mathcal{M},(2,1)}$, concretely realized as (the dual of) Lurie's relative nerve [Lur09, Definition 3.2.5.2]. The functor Φ is induced by the functors $\{\mathbf{Ind}_\kappa^S(L(-))_b\}_\kappa$, and Ψ is the forgetful functor. As in the previous paragraph, the functor Ψ is already a localization, so it suffices to verify that Φ induces a localization upon localizing at the maps whose images in $\mathcal{P}r\mathcal{S}\mathcal{M}$ are equivalences. This follows from (the dual of) [Araa, Corollary B.6] and Proposition 4.6, which show more strongly that it induces a localization when localizing at fiberwise local equivalences. The proof is now complete. \square

5. PROOF OF PROPOSITION 2.7

We finally turn to the proof of Proposition 2.7. In light of Propositions 2.5 and 2.6, it will suffice to prove this for one *specific* choice of S . To this end, we will construct a new model category of *symmetric cubical sets* in Subsection 5.1. We then use this to prove Proposition 2.7 in Subsection 5.2.

5.1. Symmetric cubical sets. In this subsection, we construct the symmetric monoidal model category of symmetric cubical sets (Theorem 5.10). It models the homotopy theory of spaces, in the sense that its underlying symmetric monoidal ∞ -category is equivalent to the cartesian monoidal ∞ -category of spaces. We will see that every combinatorial symmetric monoidal model category with cofibrant unit can be enriched over symmetric cubical sets in an essentially unique manner (Corollary 5.12).

Remark 5.1. Part of the contents of this subsection is similar in spirit to Isaacson's papers [Isa11, Isa09]. In these papers, he considered symmetric cubical sets by using a slightly bigger category of symmetric cubes. He then showed that every combinatorial symmetric monoidal model category with cofibrant unit and satisfying the monoid axiom can be enriched over symmetric cubical sets [Isa09, Theorem 10.1]. However, his approach does not seem to give an essential uniqueness of this enrichment.

We start by recalling the classical cube category and the Grothendieck model structure.

Definition 5.2. The **box category** \square has the following descriptions:

- Objects are the posets $[1]^n$, where $n \geq 0$.
- A morphism $f: [1]^n \rightarrow [1]^m$ is a poset map that erases coordinates and inserts 0 and 1 without changing the order of the coordinates. Equivalently, they are the generated under composition by the following maps:
 - (1) The **face map** $\delta^{i,\varepsilon} = \delta_n^{i,\varepsilon}: [1]^n \rightarrow [1]^{n+1}$ for $n \geq 0$ and $1 \leq i \leq n+1$, which inserts $\varepsilon \in \{0, 1\}$ in the i th coordinate.
 - (2) The **degeneracy map** $\sigma_n^i: [1]^{n+1} \rightarrow [1]^n$ for $n \geq 0$ and $1 \leq i \leq n+1$, which deletes the i th coordinate.

The cartesian product of posets makes \square into a monoidal category. It is also naturally a Reedy category. The category $\text{Set}^{\square^{\text{op}}}$ of **cubical sets** of admits an induced monoidal structure, given by Day's convolution. We will write $\square_{\leq 1} \subset \square$ for the full subcategory spanned by the objects $[1]^n$ with $n \leq 1$. The presheaf represented by $[1]^n$ will be denote by \square^n .

The category $\text{Set}^{\square^{\text{op}}}$ can be endowed with a model structure which models the homotopy theory of spaces. To state it more precisely, let $\partial \square^n = \bigcup_{i,\varepsilon} \delta^{i,\varepsilon}(\square^{n-1})$ and $\square_{i,\varepsilon}^n = \bigcup_{(j,\eta) \neq (i,\varepsilon)} \delta^{j,\varepsilon}(\square^{n-1})$. We consider the following sets of morphisms of cubical sets:

$$I = \{\partial \square^n \rightarrow \square^n \mid n \geq 0\}, \quad J = \{\square_{i,\varepsilon}^n \rightarrow \square^n \mid n \geq 1, 1 \leq i \leq n, \varepsilon \in \{0, 1\}\}.$$

Theorem 5.3. [Cis06], [Jar06, Theorem 6.2, Theorem 8.6, Theorem 8.8] *The category $\text{Set}^{\square^{\text{op}}}$ of cubical sets has a combinatorial monoidal model structure called the **Grothendieck model structure**, whose:*

- *cofibrations are the monomorphisms.*
- *weak equivalences are preserved and detected by the triangulation functor $\text{Set}^{\square^{\text{op}}} \rightarrow \text{Set}^{\Delta^{\text{op}}}$, which is the left adjoint carrying $\square^n \mapsto (\Delta^1)^n$.*

Moreover, I and J cofibrantly generate this model structure, and T is a left Quillen equivalence with respect to the Kan–Quillen model structure on simplicial sets.

Our goal for now is to consider the symmetric monoidal version of this.

Definition 5.4. We define the symmetric monoidal category of **symmetric box category** \square_{Σ} by adjoining to the box category the permutation map $\pi_p: [1]^n \rightarrow [1]^n$ for each $n \geq 0$ and $p \in \Sigma_n$, defined by $\pi_p(x_1, \dots, x_n) = (x_{p^{-1}(1)}, \dots, x_{p^{-1}(n)})$. There is an inclusion $i: \square \rightarrow \square_{\Sigma}$. We denote by \square_{Σ}^n the presheaves on \square_{Σ} represented by $[1]^n$. Note that $\square_{\leq 1}$ is a full subcategory of \square_{Σ} .

Notation 5.5. The following proposition is immediate from the definitions:

Proposition 5.6. *We have the following cocubical relations for maps in \square_{Σ} :*

- $\delta^{j,\eta} \delta^{i,\varepsilon} = \delta^{i,\varepsilon} \delta^{j-1,\eta}$ for $j \leq i$.
- $\sigma^j \delta^{i,\varepsilon} = \begin{cases} \delta^{i,\varepsilon} \sigma^{j-1} & \text{if } i < j, \\ \text{id} & \text{if } i = j, \\ \delta^{i-1} \sigma^j & \text{if } i > j. \end{cases}$
- $\sigma^j \sigma^i = \sigma^i \sigma^{j+1}$ if $i \leq j$.
- $\pi_p \pi_q = \pi_{pq}$.
- $\pi_p \delta^{i,\varepsilon} = \delta^{p(i)} \pi_q$, where q denotes the composite

$$\{1, \dots, n\} \cong \{1, \dots, n+1\} \setminus \{i\} \xrightarrow{p} \{1, \dots, n+1\} \setminus \{p(i)\} \cong \{1, \dots, n\}.$$

$$\bullet \quad \sigma^i \pi_p = \pi_q \sigma^{p^{-1}(i)}, \text{ where } q \text{ denotes the composite}$$

$$\{1, \dots, n-1\} \cong \{1, \dots, n\} \setminus \{p^{-1}(i)\} \xrightarrow{p} \{1, \dots, n\} \setminus \{i\} \cong \{1, \dots, n-1\}.$$

Moreover, every map in \square_{Σ} factors uniquely as

$$\delta^{i_1, \varepsilon_1} \dots \delta^{i_n, \varepsilon_n} \pi_p \sigma_l^{j_1} \dots \sigma^{j_m},$$

with $i_1 > \dots > i_n$, $j_1 < \dots < j_m$, and $p \in \Sigma_l$.

Corollary 5.7. *The symmetric cube category \square_{Σ} is generated by the face maps, degeneracy maps, and permutation maps under the cocubical relations. A similar claim holds for the classical cube category.*

Corollary 5.8. *Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a symmetric monoidal category. The evaluation at the unit object $[1] \in \square_{\Sigma}$ determines a categorical equivalence*

$$\theta: \text{Fun}^{\otimes}(\square_{\Sigma}, \mathcal{C}) \xrightarrow{\sim} \text{Fun}(\square_{\leq 1}, \mathcal{C}) \times_{\text{Fun}(\{[1]^0\}, \mathcal{C})} \{\mathbf{1}\}.$$

A similar claim holds for monoidal categories and the plain cube category.

Proof. The functor θ needs a bit of explanation. If $F: \square_{\Sigma} \rightarrow \mathcal{C}$ is a symmetric monoidal functor, then its image in $\text{Fun}(\square_{\leq 1}, \mathcal{C}) \times_{\text{Fun}(\{[1]^0\}, \mathcal{C})} \{\mathbf{1}\}$ is given by the functor $\square_{\leq 1} \rightarrow \mathcal{C}$ obtained by modifying the value of F at $[1]^0$ to $\mathbf{1}$ using the structure map $\mathbf{1} \xrightarrow{\cong} F([1]^0)$ of F .

With this in mind, we will give an explicit inverse equivalence to θ . Given a functor $F: \square_{\leq 1} \rightarrow \mathcal{C}$ carrying $[1]^0$ to the unit object, we can define a new functor $\tilde{F}: \square_{\Sigma} \rightarrow \mathcal{C}$ by $\tilde{F}([1]^n) = F([1])^{\otimes n}$, with structure maps given by that of F and the braiding of \mathcal{C} (Corollary 5.7). The coherence maps of \mathcal{C} makes \tilde{F} into a symmetric monoidal functor. The functor $F \mapsto \tilde{F}$ is an inverse equivalence of θ . \square

Corollary 5.9. *Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a cocomplete symmetric monoidal category whose tensor product preserves small colimits in each variable. The evaluation at the unit object $\square_{\Sigma}^0 \in \text{Set}^{\square_{\Sigma}^{\text{op}}}$ induces a categorical equivalence*

$$\text{Fun}^{\otimes, cc}(\text{Set}^{\square_{\Sigma}^{\text{op}}}, \mathcal{C}) \xrightarrow{\sim} \text{Fun}(\square_{\leq 1}, \mathcal{C}) \times_{\text{Fun}(\{[1]^0\}, \mathcal{C})} \{\mathbf{1}\},$$

where $\text{Fun}^{\otimes, cc}(\text{Set}^{\square_{\Sigma}^{\text{op}}}, \mathcal{C})$ denotes the category of symmetric monoidal categories whose underlying functors preserve small colimits, and symmetric monoidal natural transformations between them.

Proof. This follows from Corollary 5.8 and the universal property of Day convolution product monoidal structure. \square

For the following theorem, we will write $\partial \square_{\Sigma}^n \subset \square_{\Sigma}^n$ for the subpresheaf consisting of the maps $[1]^k \rightarrow [1]^n$ that inserts at least one 0 or 1. Equivalently, we have $i_!(\partial \square^n) = \partial \square_{\Sigma}^n$.

Theorem 5.10. *The category $\text{Set}^{\square_{\Sigma}^{\text{op}}}$ has a tractable model structure whose weak equivalences and fibrations are preserved and detected by the forgetful functor $i^*: \text{Set}^{\square_{\Sigma}^{\text{op}}} \rightarrow \text{Set}^{\square_{\Sigma}^{\text{op}}}$. Moreover:*

(I) *The adjunction*

$$i_!: \text{Set}^{\square_{\Sigma}^{\text{op}}} \rightleftarrows \text{Set}^{\square_{\Sigma}^{\text{op}}} : i^*$$

is a Quillen equivalence.

(II) *The model structure on $\text{Set}^{\square_{\Sigma}^{\text{op}}}$ is symmetric monoidal with respect to the Day convolution.*

(III) *The model structure is left proper.*

(IV) *If \mathbf{M} is a symmetric monoidal model category and $F: \text{Set}^{\square_{\Sigma}^{\text{op}}} \rightarrow \mathbf{M}$ is a symmetric monoidal functor which is also a left adjoint, then F is left Quillen precisely when the map $F(\partial \square_{\Sigma}^1 \rightarrow \square_{\Sigma}^1)$ is a cofibration, the map $F(\square_{\Sigma}^1) \rightarrow F(\square_{\Sigma}^0)$ is a weak equivalence, and the monoidal unit of \mathbf{M} is cofibrant.*

(V) *The assignment $\square_{\Sigma}^n \mapsto (\Delta^1)^n$ determines a symmetric monoidal left Quillen equivalence $\text{Set}^{\square_{\Sigma}^{\text{op}}} \rightarrow \text{Set}^{\Delta^{\text{op}}}$.*

The proof of Theorem 5.10 relies on the following lemma.

Lemma 5.11. *The diagram*

$$\begin{array}{ccc} \mathbf{Set}^{\square^{\text{op}}} & \xhookrightarrow{\quad} & \mathcal{S}^{\square^{\text{op}}} \\ \downarrow & & \downarrow \text{colim}_{\square^{\text{op}}} \\ \mathbf{Set}^{\square^{\text{op}}}[w\text{eq}^{-1}] & \xrightarrow{\simeq} & \mathcal{S} \end{array}$$

commutes up to natural equivalence.

Proof. We first give a model for $\text{colim}_{\square^{\text{op}}}$. Consider the functor

$$\Phi: \text{Fun}\left(\square^{\text{op}}, \mathbf{Set}^{\square^{\text{op}}}\right) \rightarrow \mathbf{Set}^{\square^{\text{op}}}, Y \mapsto \int^{[1]^n \in \square} Y_n \otimes \square^n.$$

Since the cocubical object $\square^\bullet \in \text{Fun}\left(\square, \mathbf{Set}^{\square^{\text{op}}}\right)$ is Reedy cofibrant, the functor Φ is left Quillen for the Reedy model structure [Lur09, Proposition A.2.9.26]. The right adjoint of this functor, given by $K \mapsto K^{\square^\bullet}$, is weakly equivalent to that of the diagonal functor on the full subcategory of fibrant objects. So Φ is a model of the homotopy colimit, in the sense that the induced functor functor

$$\mathcal{S}^{\square^{\text{op}}} \rightarrow \mathcal{S}$$

is equivalent to $\text{colim}_{\square^{\text{op}}}$.

Now since every cubical set X is Reedy cofibrant when regarded as a level-wise discrete cubical object in $\mathbf{Set}^{\square^{\text{op}}}$, the above argument shows that the functor $\Phi': \mathbf{Set}^{\square^{\text{op}}} \rightarrow \mathbf{Set}^{\square^{\text{op}}}, X \mapsto \Phi(X)$ descends to the composite

$$\mathbf{Set}^{\square^{\text{op}}} \rightarrow \mathcal{S}^{\square^{\text{op}}} \rightarrow \mathcal{S}.$$

The coYoneda lemma shows that Φ' is naturally isomorphic to the identity functor, and the claim follows. \square

Proof of Theorem 5.10. We start by showing that the model structure on $\mathbf{Set}^{\square^{\text{op}}}$ transfers to a model structure on $\mathbf{Set}^{\square_\Sigma^{\text{op}}}$, using [Hir03, Theorem 11.3.2]. Let $I = \{\partial \square^n \rightarrow \square^n \mid n \geq 0\}$ and $J = \{\square_{i,\varepsilon}^n \rightarrow \square^n \mid n \geq 1, 1 \leq i \leq n, \varepsilon \in \{0, 1\}\}$ denote the generating cofibrations and trivial cofibrations of $\mathbf{Set}^{\square^{\text{op}}}$. We wish to show that i^* takes $i_! J$ -cell complexes to weak equivalences. For this, it will suffice to show that each of the map in $i^* i_! J$ is a trivial cofibration. The cofibration part is clear, because $i_!$ and i^* preserve monomorphisms. Consequently, it will suffice to show that each of the maps in $i^* i_! J$ is a weak equivalence. We will prove this by showing that the unit $\eta: \text{id} \rightarrow i^* i_!$ is a natural weak equivalence.

By the standard argument using the skeletal filtration, it will suffice to show that $\eta_{\square^n}: \square^n \rightarrow i^* i_! \square^n$ is a weak equivalence for all n . In other words, our task is to prove that each $i^* i_! \square^n$ is weakly contractible. By Lemma 5.11 and [Lur09, Corollary 3.3.4.6], this is equivalent to the condition that the category $\square_{/i^* i_! \square^n} \cong \square \times_{\square_\Sigma} \square_{\Sigma/\square^n}$ be weakly contractible for all n . Thus, it suffices to show that the functor $i: \square \rightarrow \square_\Sigma$ is homotopy initial. We prove this as follows: Consider the commutative diagram

$$\begin{array}{ccc} & \square_+ & \\ j \swarrow & & \searrow k \\ \square & \xrightarrow{i} & \square_\Sigma \end{array}$$

where $\square_+ \subset \square$ denotes the subcategory spanned by the face maps. Since homotopy initial functors have the right cancellation property [Cis19b, Corollary 4.1.9], it will suffice to show that j and k are homotopy initial. We will show that j is homotopy initial; the proof for k is similar. Our task is to show that the category $\square_+ \times_{\square} \square_{/\square^n}$ is weakly contractible for every $n \geq 0$. But the inclusion $\square_+ \times_{\square} \square_{/\square^n} \hookrightarrow \square_{/\square^n}$

admits a homotopy inverse, which factors a map as a composition of degeneracy maps followed by a composition of face maps. So we are reduced to showing that $\square_{/\square^n}$ is weakly contractible, which is clear.

We now prove (I) through (V). For (I), observe that we have just shown that the derived unit is an isomorphism. Since $\mathbb{R}i^*$ is conservative by construction, the triangle identities show that the derived counit is also an isomorphism. Hence the adjunction is a Quillen equivalence. Part (II) follows from the fact that the model structure on $\mathbf{Set}^{\square^{\text{op}}}$ is monoidal for the Day convolution. Part (III) follows from the left properness of $\mathbf{Set}^{\square^{\text{op}}}$, since i^* preserves cofibrations and small colimits and detects weak equivalences. Part (IV) is proved exactly as in [Law17, Corollary 1.5]. Finally, part (V) follows from parts (I, IV) and Theorem 5.3. \square

Corollary 5.12. *Every cofibrantly generated symmetric monoidal model category \mathbf{M} with a cofibrant unit is a symmetric $\mathbf{Set}^{\square^{\text{op}}}_{\Sigma}$ -algebra in an essentially unique way. More precisely, the category $\text{Fun}^{\otimes, LQ}(\mathbf{Set}^{\square^{\text{op}}}_{\Sigma}, \mathbf{M})$ is weakly contractible.*

Proof. Define a full subcategory \mathcal{X} of

$$\text{Fun}(\square_{\leq 1}, \mathbf{M})_{\mathbf{1}} = \text{Fun}(\square_{\leq 1}, \mathbf{M}) \times_{\text{Fun}(\{[1]^0\}, \mathbf{M})} \{\mathbf{1}\}$$

as follows: An object of $F \in \text{Fun}(\square_{\leq 1}, \mathbf{M})_{\mathbf{1}}$ can be identified with a diagram of the form

$$\begin{array}{ccc} \mathbf{1} & & \mathbf{1} \\ & \searrow \text{id} & \swarrow \\ & F([1]) & \\ & \uparrow & \downarrow \\ \mathbf{1} & & \mathbf{1} \end{array}$$

We declare that such a diagram belongs to \mathcal{X} if and only if the map $\mathbf{1} \amalg \mathbf{1} \rightarrow F([1])$ is a cofibration, and the map $F([1]) \rightarrow \mathbf{1}$ is a weak equivalence.

According to Theorem 5.10 and Corollary 5.9, there is an equivalence of categories

$$\text{Fun}^{\otimes, LQ}(\mathbf{Set}^{\square^{\text{op}}}_{\Sigma}, \mathbf{M}) \xrightarrow{\sim} \mathcal{X}.$$

Therefore, it suffices to show that \mathcal{X} is weakly contractible. To this end, we define a functor $\Phi: \text{Fun}(\square_{\leq 1}, \mathbf{M})_{\mathbf{1}} \rightarrow \text{Fun}(\square_{\leq 1}, \mathbf{M})_{\mathbf{1}}$ as follows: It carries an object $(\mathbf{1} \amalg \mathbf{1} \rightarrow I \rightarrow \mathbf{1})$ to an object $(\mathbf{1} \amalg \mathbf{1} \rightarrow I' \rightarrow \mathbf{1})$, where I' is obtained by (functorially) factoring the map $\mathbf{1} \amalg \mathbf{1} \rightarrow I$ as a cofibration $\mathbf{1} \amalg \mathbf{1} \rightarrow I'$ followed by a weak equivalence $I' \xrightarrow{\sim} I$. Then Φ restricts to a functor $\Phi_{\mathcal{X}}: \mathcal{X} \rightarrow \mathcal{X}$. The functor $\Phi_{\mathcal{X}}$ admits natural transformations to the identity functor and the constant functor at $\Phi(\mathbf{1} \amalg \mathbf{1} \rightarrow \mathbf{1} \rightarrow \mathbf{1})$. Therefore, the identity functor of \mathcal{X} can be connected by a zig-zag of natural transformation to a constant functor. Hence \mathcal{X} is weakly contractible, as claimed. \square

5.2. Proof of Proposition 2.7. We can now give a proof of Proposition 2.7.

Proof of Proposition 2.7. As discussed at the beginning of the section, it will suffice to prove the claim for a specific choice of \mathbf{S} .

Take $\mathbf{S} = \mathbf{Set}^{\square^{\text{op}}}_{\Sigma}$. The functor $U: \mathbf{CombSymS-Alg} \rightarrow \mathbf{CombSMMC}^{\mathbf{1}}$ is a cocartesian fibration, and its fibers are weakly contractible by Corollary 5.12. It follows from [Lur25, Tag 02LY] that U is already a localization, so in particular it induces an equivalence

$$\mathbf{CombSymS-Alg}[\text{Quill.eq}^{-1}] \xrightarrow{\sim} \mathbf{CombSMMC}^{\mathbf{1}}[\text{Quill.eq}^{-1}].$$

\square

6. VARIATIONS OF THE MAIN THEOREM

Notation 6.1. Throughout this section, we fix a left proper tractable symmetric monoidal model category \mathbf{S} with a cofibrant unit, whose underlying symmetric monoidal ∞ -category is equivalent to the cartesian monoidal ∞ -category \mathcal{S} of ∞ -groupoids.

In Section 1, we introduced many variations of categories of symmetric monoidal model categories. In this section, we will identify the localization of most of them. We also identify the localization of their subcategories of left Quillen equivalences.

This section has two main results, one in the symmetric monoidal case (Theorem 6.2), and the other in the non-symmetric monoidal case (Theorem 6.4). Let us start by stating the symmetric monoidal case.

Theorem 6.2. *Consider the diagram of categories introduced in Section 1:*

$$\begin{array}{ccccc}
 & & \text{TractSMMC}_{\mathbf{S}, \text{semi}} & & \\
 & \nearrow & & \searrow & \\
 \text{TractSMMC}_{\mathbf{S}} & \longrightarrow & \text{TractSMMC} & \longrightarrow & \text{TractSMMC}_{\text{semi}} \\
 \downarrow & & \downarrow & & \\
 \text{CombSMMC}_{\mathbf{S}} & \longrightarrow & \text{CombSMMC} & &
 \end{array}$$

- (1) All of the vertices in the diagram localize to PrSM via the underlying symmetric monoidal ∞ -category functor.
- (2) Assertion (1) remains valid if we replace each vertex by the full subcategory consisting of those objects with a cofibrant unit.
- (3) Consider the subcategories of each vertex in the diagram spanned by Quillen equivalences. They localize to the maximal sub ∞ -groupoid $\text{PrSM}^{\simeq} \subset \text{PrSM}$ via the underlying symmetric monoidal ∞ -category functor.
- (4) Assertion (3) remains valid if we replace each vertex by the subcategory consisting of those objects with a cofibrant unit and Quillen equivalences between them.

Remark 6.3. Informally, part (3) of Theorem 6.2 asserts the following:

- presentably symmetric monoidal ∞ -categories are presented by combinatorial symmetric monoidal model categories;
- such presentations are unique up to zig-zags of Quillen equivalences;
- such zig-zags are themselves unique; and so on.

More precisely, for every $\mathcal{C} \in \text{PrSM}$, the ∞ -category

$$\text{CombSMMC}^{QE} \times_{\text{PrSM}^{\simeq}} (\text{PrSM}^{\simeq})_{/\mathcal{C}}$$

is weakly contractible, where $\text{CombSMMC}^{QE} \subset \text{CombSMMC}$ denotes the subcategory of Quillen equivalences. This follows from the fact that a functor into an ∞ -groupoid is a localization if and only if it is final [Cis19a, Proposition 7.1.10].

Distressingly, some of our method does not extend to the monoidal setting. Because of this, the monoidal version of Theorem 6.2 takes the following form:

Theorem 6.4. *Consider the diagram of categories introduced in Section 1:*

$$\begin{array}{ccc}
 & \text{TractMMC}_{S,\text{semi}} & \\
 \nearrow & & \searrow \\
 \text{TractMMC}_S & \xrightarrow{\quad} & \text{TractMMC}_{\text{semi}} \\
 \downarrow & & \\
 \text{CombMMC}_S. & &
 \end{array}$$

- (1) All of the vertices in the diagram localize to PrMon via the underlying symmetric monoidal ∞ -category functor.
- (2) Assertion (1) remains valid if we replace each vertex by the full subcategory consisting of those objects with a cofibrant unit.
- (3) Consider the subcategories of each vertex in the diagram spanned by Quillen equivalences. They localize to the maximal sub ∞ -groupoid $\text{PrMon}^\simeq \subset \text{PrMon}$ via the underlying symmetric monoidal ∞ -category functor.
- (4) Assertion (3) remains valid if we replace each vertex by the subcategory consisting of those objects with a cofibrant unit and Quillen equivalences between them.

Remark 6.5. Part (3) of Theorem 6.4 implies that every tractable monoidal semi-model is Quillen equivalent to a simplicial one. This gives a partial solution to an open problem by Hovey [Hov, Problem 10], which asks whether every monoidal model category is Quillen equivalent to a simplicial one.

We now turn to the proof of these theorems. A key step in the proof is a functorial replacement of semi-monoidal model categories by simplicial ones. The basic idea is that of Dugger [Dug01b], who observed that if \mathbf{M} is nice model category, the category $s\mathbf{M}$ of simplicial objects can be made into a simplicial model category with respect to a Bousfield localization of the Reedy model structure, assuming that it exists. Unfortunately, his idea does not apply directly to our setting, so we need a bit more work.

We will start by recalling Dugger's idea. Most of the following lemma can be found in [Dug01b]. We present a slightly more streamlined proof than loc. cit.

Lemma 6.6. *Let \mathbf{M} be a tractable semi-model category. The Reedy semi-model category $s\mathbf{M}_{\text{Reedy}}$ of simplicial objects in \mathbf{M} admits a left Bousfield localization, denoted by $s\mathbf{M}_{\text{Reedy,loc}}$, with the following properties:*

- (1) *Fibrant objects of $s\mathbf{M}_{\text{Reedy,loc}}$ are the Reedy fibrant simplicial objects X such that the map $X_0 \rightarrow X_n$ is a weak equivalence for all $n \geq 0$.*
- (2) *The diagonal functor $\delta: \mathbf{M} \rightarrow s\mathbf{M}_{\text{Reedy,loc}}$ is a right Quillen equivalence.*
- (3) *$s\mathbf{M}_{\text{Reedy,loc}}$ is a \mathbf{sSet} -enriched semi-model category.*

Proof. Consider the functor

$$\left(\mathbf{M}^{\Delta^{\text{op}}}\right)_\infty \simeq (\mathbf{M}_\infty)^{\Delta^{\text{op}}} \xrightarrow{\text{colim}} \mathbf{M}_\infty,$$

where the first equivalence is that of [Cis19a, Theorem 7.9.8]. Since Δ^{op} is weakly contractible, the right adjoint of this functor is fully faithful [Lur09, Corollary 4.4.4.10]. Thus, applying Proposition B.4 (or rather, Remark B.5) to this functor, we obtain a semi-model structure $s\mathbf{M}_{\text{Reedy,loc}}$ on $s\mathbf{M}$ satisfying conditions (1) and (2).

For (3), we must describe the simplicial enrichment of $s\mathbf{M}$. If X and Y are simplicial objects in $s\mathbf{M}$, we define the mapping simplicial object $\text{Map}(X, Y)$ as follows: Given a simplicial set S , we define a new simplicial object $S \otimes X \in s\mathbf{M}$

by $(S \otimes X)_n = \coprod_{S_n} X_n$. We then define $\text{Map}(X, Y)_n = \text{Hom}_{s\mathbf{M}}(\Delta^n \otimes X, Y)$. This makes $s\mathbf{M}$ into a simplicial category.

To complete the proof, we must show that $s\mathbf{M}_{\text{Reedy,loc}}$ is an $s\text{Set}$ -enriched model category. Let $(-)^S: s\mathbf{M} \rightarrow s\mathbf{M}$ denote the right adjoint of the functor $S \otimes -$. We must prove the following:

- (i) If $i: K \rightarrow L$ is a cofibration in $s\text{Set}$ and $f: X \rightarrow Y$ is a Reedy cofibration of Reedy cofibrant objects of $s\mathbf{M}$, then the induced map

$$i \square f: K \otimes Y \amalg_{K \otimes X} L \otimes X \rightarrow L \otimes Y$$

is a Reedy cofibration.

- (ii) In the situation of (i), if further f is a weak equivalence of $s\mathbf{M}_{\text{Reedy,loc}}$, then so is $i \square f$.
- (iii) In the situation of (i), if further i is anodyne, then $i \square f$ is a weak equivalence of $s\mathbf{M}_{\text{Reedy,loc}}$.

To prove these assertions, choose generating sets I and J of cofibrations and trivial cofibrations of \mathbf{M} whose elements have a cofibrant domain. Also, for each $n \geq 0$, write $i_n: \partial\Delta^n \hookrightarrow \Delta^n$ for the inclusion. The Reedy semi-model structure on \mathbf{M} is generated by the sets $\tilde{I} = \{i_n \square i \mid n \geq 0, i \in I\}$ and $\tilde{J} = \{i_n \square j \mid n \geq 0, j \in J\}$, and assertion (i) is immediate from this.

For (ii), by two out of three it suffices to show that the maps $K \otimes f$ (and $L \otimes f$) are weak equivalences. By part (i) and [JT07, Lemma 7.14], this will follow if we show that $(-)^K: s\mathbf{M}_{\text{Reedy,loc}} \rightarrow s\mathbf{M}_{\text{Reedy,loc}}$ preserves fibrations of fibrant objects. The explicit generating set above makes it clear that $(-)^K$ carries Reedy fibrations to Reedy fibrations. Since a map of fibrant objects of $s\mathbf{M}_{\text{Reedy,loc}}$ is a fibration if and only if it is a Reedy fibration [Hir03, Proposition 3.3.16], we are therefore reduced to showing that $(-)^K$ preserves fibrant objects of $s\mathbf{M}_{\text{Reedy,loc}}$.

Let $X \in s\mathbf{M}_{\text{Reedy,loc}}$ be fibrant. We already know that X^K is Reedy fibrant, so it suffices to show that the map $\theta: (X^K)_0 \rightarrow (X^K)_n$ is a weak equivalence for all n . For this, set $\text{hom}(S, X) = \lim_{\Delta^n \rightarrow S \in (\mathbf{A}_{/S})^{\text{op}}} X_n$ for each simplicial set S . We can then identify θ with the map $\text{hom}(K \times \Delta^0, X) \rightarrow \text{hom}(K \times \Delta^n, X)$. To prove that this is a weak equivalence, it suffices to show that the map

$$\text{hom}(\Delta^0, X) \rightarrow \text{hom}(\Delta^n, X)$$

is a weak equivalence (see the argument of [Cis19a, Proposition 4.6.1]). This follows from our assumption on X .

Finally, for (iii), by [JT07, Lemma 7.14], it suffices to show that for each fibration $p: E \rightarrow B$ of fibrant objects, the map

$$i \pitchfork p: E^L \rightarrow E^K \times_B B^L$$

is a trivial fibration. We already know from (i) that this is a fibration, so it suffices to show that the map $E^L \rightarrow E^K$ and $B^L \rightarrow B^K$ are weak equivalences. Using (the argument of) [Cis19a, Proposition 4.6.1], it will suffice to prove this in the case where i is a map of the form $\Delta^0 \rightarrow \Delta^n$. In this case, the claim is clear, because E and B are assumed to be fibrant in $s\mathbf{M}_{\text{Reedy,loc}}$. \square

Unfortunately, when \mathbf{M} is a monoidal semi-model category, it seems difficult to compare $s\mathbf{M}_{\text{Reedy,loc}}$ with \mathbf{M} via a monoidal left Quillen functor. An obvious comparison map would be the diagonal functor $\mathbf{M} \rightarrow s\mathbf{M}$, but it is rarely left Quillen. So we need to increase the class of cofibrations of to force this to be true, and this is what we are going to do.

To state the next lemma, it will be useful to introduce the following terminology.

Definition 6.7. We say that a semi-model category \mathbf{M} is κ -tractable if it satisfies the following pair of conditions:

- (1) The underlying category of \mathbf{M} is locally κ -presentable.
- (2) The (essentially small) class of cofibrations of κ -compact cofibrant objects of \mathbf{M} generates the class of cofibrations of \mathbf{M} .

Lemma 6.8. Let κ be an uncountable regular cardinal, and let \mathbf{M} be a κ -tractable semi-model category. There is a κ -tractable semi-model structure on $s\mathbf{M}$, denoted by $s\mathbf{M}_{\kappa\text{-inj,loc}}$ with the following properties:

- (1) Cofibrations are generated by the maps $X \rightarrow Y$ such that, for each $n \geq 0$, the map $X_n \rightarrow Y_n$ is a cofibration of κ -compact cofibrant objects of \mathbf{M} .
- (2) Weak equivalences are the hocolim-equivalences, i.e., maps inducing weak equivalences between the homotopy colimits.
- (3) The diagonal functor $\delta: \mathbf{M} \rightarrow s\mathbf{M}_{\text{inj,loc}}$ is a left Quillen equivalence.
- (4) $s\mathbf{M}_{\text{inj,loc}}$ is a $s\text{Set}$ -enriched semi-model category.
- (5) If \mathbf{M} is a monoidal semi-model category, then $s\mathbf{M}_{\text{inj,loc}}$ is also one for the degreewise tensor product.

Proof. We first construct a κ -tractable semi-model structure satisfying conditions (1) through (3) [BW24, Theorem B]. (Note that, since κ is uncountable, κ -compact objects of $s\mathbf{M}$ are simply the levelwise κ -compact simplicial objects [Low13, Proposition 2.23].) Call a map $X \rightarrow Y$ of $s\mathbf{M}$ an **κ -injective cofibration** if it is a cofibration in the sense of condition (1). We also say that a map of $s\mathbf{M}$ is a **κ -injective fibration** if it has the right lifting property for κ -injective cofibrations. We must verify the following:

- (i) The full subcategory of $\text{Fun}([1], s\mathbf{M})$ spanned by the hocolim-equivalences is an accessible subcategory.⁵
- (ii) Every κ -injective fibration is a hocolim-equivalences.
- (iii) Given a pushout diagram

$$\begin{array}{ccc} X & \longrightarrow & X' \\ i \downarrow & & \downarrow i' \\ Y & \longrightarrow & Y' \end{array}$$

of κ -injectively cofibrant objects, if i is a hocolim-equivalence and a κ -injective cofibration, so is i' .

For part (i), by [Lur09, Proposition 5.4.6.6], it suffices to show that

$$\mathbf{M}^{\Delta^{\text{op}}} \rightarrow \left(\mathbf{M}^{\Delta^{\text{op}}} \right)_{\infty} \simeq (\mathbf{M}_{\infty})^{\Delta^{\text{op}}} \xrightarrow{\text{colim}} \mathbf{M}_{\infty}$$

is accessible. The functor colim is clearly accessible (being a left adjoint), so it suffices to show that the first functor is accessible. This follows from the fact that for a sufficiently large regular cardinal κ , weak equivalences of $\mathbf{M}^{\Delta^{\text{op}}}$ are stable under κ -filtered colimits in $\mathbf{M}^{\Delta^{\text{op}}}$ (see [RR15, Proposition 4.1]).

For part (ii), let $p: X \rightarrow Y$ be a κ -injective fibration of $s\mathbf{M}$. We claim that p is levelwise a weak equivalence (hence a hocolim-equivalence). Since \mathbf{M} is κ -tractable, it suffices to show that every lifting problem as depicted in the left-hand diagram below admits a solution, provided that i is a cofibration of κ -compact objects of

⁵In [BW24], it is required that weak equivalences with cofibrant domains span an accessible full subcategory. However, the proof goes through without cofibrancy assumption, as verified via private communication with David White.

M:

$$\begin{array}{ccc} A & \longrightarrow & X_n \\ i \downarrow & \nearrow & \downarrow p \\ B & \longrightarrow & Y_n \end{array} \quad \begin{array}{ccc} \Delta^n \otimes \delta(A) & \longrightarrow & X \\ \Delta^n \otimes \delta(i) \downarrow & \nearrow & \downarrow p \\ \Delta^n \otimes \delta(B) & \longrightarrow & Y. \end{array}$$

The lifting problem is equivalent to the one on the right, which is solvable because $\Delta^n \otimes \delta(i)$ is a κ -injective cofibration.

Par (iii) is clear, because such squares are homotopy cocartesian in $\mathbf{M}^{\Delta^{\text{op}}}$ (as it is levelwise homotopy cocartesian).

Next, we prove (4). Let $i: K \rightarrow L$ be a monomorphism of simplicial sets, and let $f: X \rightarrow Y$ be a generating κ -injective cofibration $s\mathbf{M}_{\kappa\text{-inj,loc}}$. We must show that the map

$$i \square f: (K \otimes Y) \amalg_{K \otimes X} (L \otimes X) \rightarrow L \otimes Y$$

is a κ -injective cofibration cofibration, and that it is a weak equivalence if either i or f is one. For the claim on cofibration, it suffices to prove this in the case where i is the boundary inclusion $\partial\Delta^n \subset \Delta^n$ for some $n \geq 0$. In this case, $i \square f$ is another generating κ -injective cofibration, so there is nothing to prove. For the acyclic part of the claim, find a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ p \downarrow & & \downarrow q \\ X & \xrightarrow{f} & Y \end{array}$$

where p and q are trivial Reedy fibrations, X' is Reedy cofibrant, and f' is a Reedy cofibration. We then have a diagram

$$\begin{array}{ccc} (K \otimes Y') \amalg_{K \otimes X'} (L \otimes X') & \longrightarrow & L \otimes Y' \\ \downarrow & & \downarrow \\ (K \otimes Y) \amalg_{K \otimes X} (L \otimes X) & \longrightarrow & L \otimes Y \end{array}$$

whose vertical arrows are levelwise weak equivalences (because the pushouts are levelwise homotopy pushouts). Thus, it suffices to show that $i \square f'$ is a weak equivalence. This follows from Lemma 6.6.

Finally, we prove (5). Suppose that \mathbf{M} is a monoidal model category. We must show that $s\mathbf{M}_{\kappa\text{-inj,loc}}$ is also a monoidal model category. The only nontrivial part is the acyclic part of the pushout-product axiom. For this, it suffices to show that for each cofibrant $X \in s\mathbf{M}_{\kappa\text{-inj,loc}}$, the functors $X \otimes -$, $- \otimes X: s\mathbf{M}_{\kappa\text{-inj,loc}} \rightarrow s\mathbf{M}_{\kappa\text{-inj,loc}}$ preserve weak equivalences of cofibrant objects. This follows from the fact that Δ^{op} is a sifted ∞ -category [Lur09, Lemma 5.5.8.4], so that $\text{hocolim}_{\Delta^{\text{op}}}(X \otimes Y) \simeq (\text{hocolim}_{\Delta^{\text{op}}} X) \otimes (\text{hocolim}_{\Delta^{\text{op}}} Y)$ whenever $X, Y \in s\mathbf{M}$ are levelwise cofibrant. The proof is now complete. \square

Proof of Theorems 6.2 and 6.4. We consider the symmetric monoidal case; the monoidal case is similar. We start with Part (1). We treat each case separately:

- **CombSMMC:** This is the content of Theorem 2.4.
- **TractSMMC:** The proof of Theorem 2.4 goes through by replacing **CombSMMC** by **TractSMMC**.
- **TractSMMCs and CombSMMCs and TractSMMCs_{semi}:** The proof of Proposition 2.6 goes through by replacing **CombSymS-Alg** by these categories. (In the semi-case, we only need **S** to be a left proper tractable symmetric monoidal semi-model category.)

- $\text{TractSMMC}_{\text{semi}}$: For each uncountable regular cardinal κ , we let $\text{TractSMMC}_{\text{semi}}(\kappa) \subset \text{TractSMMC}_{\text{semi}}$ denote the subcategory spanned by the objects whose underlying semi-model categories are κ -tractable, and the morphisms that preserve κ -compact objects. We define $\text{TractSMMC}_{\text{sSet,semi}}(\kappa)$ similarly. Lemma 6.8 gives us a functor

$$\int^{\text{urCard}} \text{TractSMMC}_{\text{semi}}(-) \rightarrow \text{TractSMMC}_{\text{sSet,semi}}(\kappa),$$

$$(\kappa, \mathbf{M}) \mapsto s\mathbf{M}_{\kappa\text{-inj,loc}},$$

which fits into the (non-commutative) diagram

$$\begin{array}{ccc} \int^{\text{urCard}} \text{TractSMMC}_{\text{sSet,semi}}(-) & \xrightarrow{U} & \int^{\text{urCard}} \text{TractSMMC}_{\text{semi}}(-) \\ \pi \downarrow & \swarrow \Phi & \downarrow \pi' \\ \text{TractSMMC}_{\text{sSet,semi}} & \xrightarrow{U'} & \text{TractSMMC}_{\text{semi}}. \end{array}$$

Here π, π', U, U' are all forgetful functors. Since $\text{TractSMMC}_{\text{sSet,semi}}$ is the colimit of $\text{TractSMMC}_{\text{sSet,semi}}(\kappa)$ as κ ranges over urCard , the functor π and π' are localizations [Lur25, Tag 02UU]. Thus, it suffices to show that the diagram commutes up to equivalence when we localize these categories by Quillen equivalences.

By Lemma 6.8, the lower triangle commutes up to natural Quillen equivalence. For the upper triangle, recall from the previous bullet point that the composite

$$\text{TractSMMC}_{\text{sSet,semi}} \xrightarrow{U'} \text{TractSMMC}_{\text{semi}} \rightarrow \mathcal{P}r\mathcal{S}\mathcal{M}$$

is a localization at Quillen equivalences. We also have a natural Quillen equivalence $U'\pi \xrightarrow{\sim} U'\Phi U$ by Lemma 6.8, so this proves the claim on the lower triangle.

Part (2) follows from part (1) and Muro's theorem [Mur15, Theorem 1] or its enriched version (Proposition C.1). Parts (3) and (4) are proved similarly, noting that the proof of Theorem 2.4 goes through if we restrict our attention to the relevant subcategories. This completes the proof of the theorem. \square

APPENDIX A. MULTIPLICATIVE GABRIEL–ULMER DUALITY

The Gabriel–Ulmer duality [GU71] is one precise formulation of the slogan that “locally presentable categories are controlled by small data.” In the ∞ -categorical setting, it asserts that for every regular cardinal κ , the Ind_κ -completion functor

$$\text{Ind}_\kappa: \mathcal{C}\mathcal{A}\mathcal{T}_\infty^{\text{idem}}(\kappa) \rightarrow \mathcal{P}r^L(\kappa)$$

is an equivalence. Here $\mathcal{C}\mathcal{A}\mathcal{T}_\infty^{\text{idem}}(\kappa)$ denotes the ∞ -category of small idempotent cocomplete ∞ -categories with κ -small colimits and functors preserving κ -small colimits, and $\mathcal{P}r^L(\kappa)$ denotes the ∞ -category of κ -presentable ∞ -categories.⁶

The goal of this section is to enhance this to an equivalence of symmetric monoidal ∞ -categories (Proposition A.5 and Corollary A.6). As a corollary to this, we obtain a Gabriel–Ulmer duality for κ -presentably monoidal ∞ -categories (Corollary A.8).

For the following definition, we recall that the ∞ -category $\mathcal{P}r^L$ admits a natural symmetric monoidal structure [Lur17, Proposition 4.8.1.15].

⁶An ∞ -category is **κ -presentable** if it is κ -accessible and presentable.

Definition A.1. Let κ be a regular cardinal. We write $\mathcal{P}r^L(\kappa)^\otimes \subset (\mathcal{P}r^L)^\otimes$ for the suboperad spanned by the κ -presentable ∞ -categories and those maps $(\mathcal{C}_1, \dots, \mathcal{C}_n) \rightarrow \mathcal{C}$ whose corresponding functors $F: \mathcal{C}_1 \times \dots \times \mathcal{C}_n \rightarrow \mathcal{C}$ have the following property:

- Given an object $(X_1, \dots, X_n) \in \mathcal{C}_1 \times \dots \times \mathcal{C}_n$, if each X_i is κ -presentable, then so is $F(X_1, \dots, X_n)$.

Given another object $\mathcal{D} \in \mathcal{P}r^L(\kappa)$, we write $\text{Fun}_{\mathcal{P}r^L(\kappa)}((\mathcal{C}_1, \dots, \mathcal{C}_n), \mathcal{D}) \subset \text{Fun}(\mathcal{C}_1 \times \dots \times \mathcal{C}_n, \mathcal{D})$ for the full subcategory spanned by the maps $(\mathcal{C}_1, \dots, \mathcal{C}_n) \rightarrow \mathcal{D}$ in $\mathcal{P}r^L(\kappa)^\otimes$.

Our goal for the moment is to show that $\mathcal{P}r^L(\kappa)^\otimes$ is a symmetric monoidal ∞ -category, and upgrade the Gabriel–Ulmar duality to a symmetric monoidal equivalence $\mathcal{P}r^L(\kappa)^\otimes$ (Proposition A.5).

Definition A.2. Let κ be a regular cardinal. We write $\mathcal{C}at_\infty^{\text{idem}}(\kappa)$ for the ∞ -category of idempotent complete small ∞ -categories with κ -small colimits. We also write $\overline{\mathcal{C}at}_\infty^{\text{idem}}(\kappa)$ for the ∞ -category of large but essentially small, idempotent complete ∞ -categories with κ -small colimits. Each of these ∞ -categories have a natural symmetric monoidal structure, as explained in [Lur17, Corollary 4.8.1.4]. (Briefly, if $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{C}at_\infty^{\text{idem}}(\kappa)$, then a giving a map $\mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{C}$ is equivalent to giving a functor $\mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$ that preserves κ -small colimits in each variable.) We denote these symmetric monoidal ∞ -categories by $\mathcal{C}at_\infty^{\text{idem}}(\kappa)^\otimes$ and $\overline{\mathcal{C}at}_\infty^{\text{idem}}(\kappa)^\otimes$. (When κ is uncountable, we drop idem from the notation because idempotent completeness is automatic.)

Our goal for the moment is to show that $\mathcal{P}r^L(\kappa)^\otimes$ is a symmetric monoidal ∞ -category, and upgrade the Gabriel–Ulmar duality to a symmetric monoidal equivalence (Proposition A.5).

Construction A.3. For each $\mathcal{C} \in \mathcal{C}at_\infty^{\text{idem}}(\kappa)$, the κ -ind completion $\text{Ind}_\kappa(\mathcal{C})$ is a κ -presentable ∞ -category and is characterized by the following universal property: For every $\mathcal{D} \in \mathcal{P}r^L(\kappa)$, the map

$$\text{Fun}_{\mathcal{P}r^L(\kappa)}(\text{Ind}_\kappa(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \text{Fun}_{\overline{\mathcal{C}at}_\infty^{\text{idem}}(\kappa)}(\mathcal{C}, \mathcal{D}_\kappa)$$

is a categorical equivalence ([Lur09, Propositions 5.3.5.10 and 5.5.1.9]). It follows that the assignment $\mathcal{C} \mapsto \text{Ind}_\kappa(\mathcal{C})$ can be assembled into a left adjoint $\text{Ind}_\kappa: \mathcal{C}at_\infty^{\text{idem}}(\kappa) \rightarrow \mathcal{P}r^L(\kappa)$. By inspecting the unit and counit, we deduce that this functor is in fact an equivalence of ∞ -categories. Using [Lur17, Remark 4.8.1.8], we find that the κ -ind completion functor $\text{Ind}_\kappa: \mathcal{C}at_\infty^{\text{idem}}(\kappa) \rightarrow \mathcal{P}r^L$ can be enhanced to a symmetric monoidal functor $\text{Ind}_\kappa: \mathcal{C}at_\infty^{\text{idem}}(\kappa)^\otimes \rightarrow (\mathcal{P}r^L)^\otimes$.

Remark A.4. The proof of [Lur09, Proposition 5.5.1.9] and [Lur09, Propositions 5.3.5.10] give a stronger universal property of $\text{Ind}_\kappa(\mathcal{C})$, where $\mathcal{C} \in \mathcal{C}at_\infty^{\text{idem}}(\kappa)$. Namely, if \mathcal{D} is an ∞ -category with small colimits, the functor

$$\text{Fun}^L(\text{Ind}_\kappa(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \text{Fun}_\kappa(\mathcal{C}, \mathcal{D})$$

is an equivalence. Here $\text{Fun}^L(\text{Ind}_\kappa(\mathcal{C}), \mathcal{D}) \subset \text{Fun}(\text{Ind}_\kappa(\mathcal{C}), \mathcal{D})$ denotes the full subcategory of functors preserving small colimits, and $\text{Fun}_\kappa(\mathcal{C}, \mathcal{D}) \subset \text{Fun}(\mathcal{C}, \mathcal{D})$ denotes the full subcategory of functors preserving κ -small colimits.

Proposition A.5. Let κ be a regular cardinal.

- (1) The ∞ -operad $\mathcal{P}r^L(\kappa)^\otimes$ is a symmetric monoidal ∞ -category, and the inclusion $\mathcal{P}r^L(\kappa)^\otimes \hookrightarrow (\mathcal{P}r^L)^\otimes$ is symmetric monoidal.

(2) The symmetric monoidal functor $\text{Ind}_\kappa: \text{Cat}_\infty^{\text{idem}}(\kappa)^\otimes \rightarrow (\mathcal{P}r^L)^\otimes$ restricts to an equivalence of symmetric monoidal ∞ -categories $\text{Cat}_\infty^{\text{idem}}(\kappa)^\otimes \xrightarrow{\sim} \mathcal{P}r^L(\kappa)^\otimes$.

Proof. We start with (1). Let $\mathcal{C}_1, \dots, \mathcal{C}_n$ be κ -presentable ∞ -categories. We wish to show that there is a κ -presentable ∞ -category \mathcal{C} and a morphism $(\mathcal{C}_1, \dots, \mathcal{C}_n) \rightarrow \mathcal{C}$ in $\mathcal{P}r^L(\kappa)^\otimes$ with the following properties:

I For every $\mathcal{D} \in \mathcal{P}r^L$, the functor

$$\theta_I: \text{Fun}_{\mathcal{P}r^L}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Fun}_{\mathcal{P}r^L}((\mathcal{C}_1, \dots, \mathcal{C}_n), \mathcal{D})$$

is an equivalence.

II For every $\mathcal{D} \in \mathcal{P}r^L(\kappa)$, the functor

$$\theta_{II}: \text{Fun}_{\mathcal{P}r^L(\kappa)}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Fun}_{\mathcal{P}r^L(\kappa)}((\mathcal{C}_1, \dots, \mathcal{C}_n), \mathcal{D})$$

is an equivalence.

To this end, choose $\mathcal{A}_i \in \text{Cat}_\infty^{\text{idem}}(\kappa)$ and equivalences $\text{Ind}_\kappa(\mathcal{A}_i) \xrightarrow{\sim} \mathcal{C}_i$, and set $\mathcal{C} = \text{Ind}_\kappa(\mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_k)$. Using Remark A.4 repeatedly, we obtain a functor $\alpha: \mathcal{C}_1 \times \dots \times \mathcal{C}_k \rightarrow \mathcal{C}$ that preserves colimits in each variable separately and which makes the diagram

$$\begin{array}{ccc} \mathcal{A}_1 \times \dots \times \mathcal{A}_n & \longrightarrow & \mathcal{C}_1 \times \dots \times \mathcal{C}_n \\ \alpha' \downarrow & & \downarrow \alpha \\ \mathcal{A} & \longrightarrow & \mathcal{C} \end{array}$$

commutative up to equivalence. The commutativity of the diagram implies that α determines a morphism $\alpha: (\mathcal{C}_1, \dots, \mathcal{C}_n) \rightarrow \mathcal{C}$ in $\mathcal{P}r^L(\kappa)^\otimes$. Remark A.4 shows that α satisfies condition (I). It follows that the functor θ_{II} in condition (II) is fully faithful. To show that it is essentially surjective, it will suffice to show that the essential image of α' generates \mathcal{A} under κ -small colimits. This follows from the proof of [Lur17, Proposition 4.8.1.3], proving (1).

Next, we prove (2). Since $\text{Ind}_\kappa: \text{Cat}_\infty^{\text{idem}}(\kappa) \rightarrow \mathcal{P}r^L(\kappa)$ is an equivalence of ∞ -categories, it suffices to show that Ind_κ carries $\text{Cat}_\infty^{\text{idem}}(\kappa)^\otimes$ into $\mathcal{P}r^L(\kappa)^\otimes$. (Because this and part (1) will prove that the resulting functor $\text{Cat}_\infty^{\text{idem}}(\kappa)^\otimes \rightarrow \mathcal{P}r^L(\kappa)^\otimes$ is symmetric monoidal.) This is equivalent to saying that $\text{Ind}_\kappa: \text{Cat}_\infty^{\text{idem}}(\kappa)^\otimes \rightarrow (\mathcal{P}r^L)^\otimes$ carries active cocartesian morphisms into $(\mathcal{P}r^L(\kappa))^\otimes$, which follows from the proof of (1). The proof is now complete. \square

We conclude this subsection with three corollaries of Proposition A.5.

Corollary A.6. *Let κ be a regular cardinal. The forgetful functor*

$$(-)_\kappa: \mathcal{P}r^L(\kappa)^\otimes \rightarrow \overline{\text{Cat}}_\infty^{\text{idem}}(\kappa)^\otimes$$

is an equivalence of symmetric monoidal ∞ -categories.

Proof. The composite $\text{Ind}_\kappa: \text{Cat}_\infty^{\text{idem}}(\kappa)^\otimes \xrightarrow{\sim} \mathcal{P}r^L(\kappa)^\otimes \rightarrow \overline{\text{Cat}}_\infty^{\text{idem}}(\kappa)^\otimes$ is equivalent to the inclusion, which is an equivalence of symmetric monoidal ∞ -categories. \square

Definition A.7. Let \mathcal{C} be a monoidal ∞ -category, and let κ be a regular cardinal. We say that \mathcal{C} is **κ -presentably monoidal** if it satisfies the following conditions:

- (1) The underlying ∞ -category of \mathcal{C} is κ -presentable.
- (2) For every $n \geq 0$ and κ -presentable objects $X_1, \dots, X_n \in \mathcal{C}$, the object $X_1 \otimes \dots \otimes X_n$ is κ -presentable. (In particular, the unit object is κ -presentable).

We write $\kappa\text{-}\mathcal{P}r\mathcal{SM} \subset \widehat{\mathcal{SM}\text{Cat}}_\infty$ for the subcategory of κ -presentably monoidal ∞ -categories and monoidal functors that preserves small colimits and κ -presentable objects.

Corollary A.8. *Let κ be a regular cardinal. The forgetful functor*

$$\kappa\text{-}\mathcal{PSM} \rightarrow \widehat{\mathcal{SMCat}}_{\infty}^{\text{idem}}(\kappa), \mathcal{C}^{\otimes} \mapsto \mathcal{C}_{\kappa}^{\otimes}$$

is a categorical equivalence, where $\widehat{\mathcal{SMCat}}_{\infty}^{\text{idem}}(\kappa) \subset \widehat{\mathcal{SMCat}}_{\infty}$ denotes the subcategory of essentially small monoidal ∞ -categories compatible with κ -small colimits, and those monoidal functors preserving κ -small colimits.

Proof. This follows from Corollary A.6 and the straightening–unstraightening equivalence. \square

Corollary A.9. *Let κ be a regular cardinal, and let $I: \widehat{\mathcal{SMCat}}_{\infty}^{\text{idem}}(\kappa) \rightarrow \kappa\text{-}\mathcal{PSM}$ be a functor equipped with a natural transformation depicted as*

$$\begin{array}{ccc} \widehat{\mathcal{SMCat}}_{\infty}^{\text{idem}}(\kappa) & \xrightarrow{I} & \kappa\text{-}\mathcal{PSM} \\ & \searrow \alpha \quad \nearrow & \\ & \widehat{\mathcal{SMCat}}_{\infty} & \end{array}$$

Suppose that, for each $\mathcal{C} \in \widehat{\mathcal{SMCat}}_{\infty}^{\text{idem}}(\kappa)$, the map $\alpha_{\mathcal{C}}: \mathcal{C} \rightarrow I(\mathcal{C})$ exhibits $I(\mathcal{C})$ as an Ind_{κ} -completion of \mathcal{C} . Then I is a categorical equivalence.

Proof. The natural transformation α determines a natural equivalence

$$\mathcal{C} \xrightarrow{\sim} I(\mathcal{C})_{\kappa}$$

from the inclusion $\iota: \widehat{\mathcal{SMCat}}_{\infty}^{\text{idem}}(\kappa) \hookrightarrow \widehat{\mathcal{SMCat}}_{\infty}(\kappa)$ to the composite $\widehat{\mathcal{SMCat}}_{\infty}^{\text{idem}}(\kappa) \xrightarrow{I} \kappa\text{-}\mathcal{PSM} \xrightarrow{(-)_{\kappa}} \widehat{\mathcal{SMCat}}_{\infty}(\kappa)$. The functors ι and $(-)^{\kappa}$ are equivalences (Corollary A.8), so we find that I is also an equivalence. \square

APPENDIX B. MISCELLANEOUS RESULTS ON LOCALIZATION

In this appendix, we record some results on localization of ∞ -categories.

B.1. Base change of localization. Let $L: \mathcal{C} \rightarrow \mathcal{D}$ be a localization of ∞ -categories, and suppose we are given a pullback square in \mathcal{Cat}_{∞}

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{D} \\ L' \downarrow & & \downarrow L \\ \mathcal{D}' & \longrightarrow & \mathcal{D}. \end{array}$$

The functor L' is not generally a localization. This naturally raises the question of when L' is actually a localization. The goal of this subsection is to record a few results related to this question.

We start with a sufficient condition for a localization functor to be stable under base change; such a functor is called a **universal localization**. (A general criteria for universal localization is discussed in [Hin24].) The following theorem says that localization of ∞ -categories with weak equivalences and fibrations [Cis19a, Definition 7.4.12] (such as that of model categories) enjoy this property:

Theorem B.1. *Let \mathcal{C} be an ∞ -category with weak equivalences and fibrations. The localization functor*

$$\mathcal{C} \rightarrow \mathcal{C}[\text{weq}^{-1}]$$

is a universal localization.

The proof of Theorem B.1 requires the following lemma:

Lemma B.2. *Let \mathcal{C} be an ∞ -category, let $\mathcal{W} \subset \mathcal{C}$ be a subcategory containing all objects, and let $L: \mathcal{C} \rightarrow \mathcal{D}$ be a functor of ∞ -categories which carries every morphism in \mathcal{W} to an equivalence. If for each $n \geq 0$, the functor*

$$\theta_{L,n}: \mathrm{Fun}([n], \mathcal{C}) \times_{\mathcal{C}^{n+1}} \mathcal{W}^{n+1} \rightarrow \mathrm{Fun}([n], \mathcal{D})^{\simeq}$$

is a weak homotopy equivalence, then L is a universal localization.

Proof. In the case where \mathcal{W} is the subcategory of morphisms mapped to equivalences in \mathcal{D} , this is more or less immediate from Mazel–Gee’s localization theorem (see [MG19, Theorem 3.8] or [AC25, Theorem 1.1]) and can be deduced from [Hin24, Lemma 1.3 and 1.5]. The proof of the general case we present below is very similar. (In fact, we do not need this generality for this paper, but we record it for completeness.)

Let $\mathcal{D}' \rightarrow \mathcal{D}$ be a categorical fibration of ∞ -categories, and set $\mathcal{C}' = \mathcal{D}' \times_{\mathcal{D}} \mathcal{C}$. We must show that the functor $L': \mathcal{C}' \rightarrow \mathcal{D}'$ is a localization. For this, let $\mathcal{W}' = (\mathcal{D}')^{\simeq} \times_{\mathcal{D}} \mathcal{W} \subset \mathcal{C}'$. We claim that L' is a localization at \mathcal{W}' . According to the generalized Mazel–Gee’s localization theorem [Ara23, Theorem 1.7], it suffices to show that for each $n \geq 0$, the map $\theta_{L',n}$ is a weak homotopy equivalence. But θ'_n is a pullback of the map θ_n along the the Kan fibration $\mathrm{Fun}([n], \mathcal{D}')^{\simeq} \rightarrow \mathrm{Fun}([n], \mathcal{D})^{\simeq}$, so the claim follows from the right properness of the Kan–Quillen model structure. \square

Proof of Theorem B.1. Let $\mathcal{W} \subset \mathcal{C}$ denote the subcategory of weak equivalences. By [Cis19a, Remark 7.5.22], we may assume that \mathcal{W} is saturated, i.e., that it consists of the morphisms whose images in $L(\mathcal{C}) = \mathcal{C}[\mathrm{weq}^{-1}]$ are equivalences. By Lemma B.2, it will suffice to show that the functor

$$\theta_{\mathcal{C},n}: \mathrm{Fun}([n], \mathcal{C}) \times_{\mathcal{C}^{n+1}} \mathcal{W}^{n+1} \rightarrow \mathrm{Fun}([n], L(\mathcal{C}))$$

is a weak homotopy equivalence for every $n \geq 0$. By [Cis19a, Theorem 7.4.20], the ∞ -category $\mathrm{Fun}([n], \mathcal{C})$ has the structure of an ∞ -category with weak equivalences and fibrations whose subcategory of weak equivalences is given by $\mathrm{Fun}([n], \mathcal{C}) \times_{\mathcal{C}^{n+1}} \mathcal{W}^{n+1}$. Moreover, by [Cis19a, Theorem 7.6.17], the functor

$$L(\mathrm{Fun}([n], \mathcal{C})) \rightarrow \mathrm{Fun}([n], L(\mathcal{C}))$$

is a categorical equivalence. Therefore, the map $\theta_{L,n}$ can be identified with $\theta_{\mathrm{Fun}([n], \mathcal{C}), 0}$. Thus, replacing \mathcal{C} by $\mathrm{Fun}([n], \mathcal{C})$, we are reduced to showing that $\theta_{\mathcal{C},0}: \mathcal{W} \rightarrow L(\mathcal{C})$ is a weak homotopy equivalence. This is the content of [Cis19a, Lemma 7.6.9], and the proof is complete. \square

We next consider the interaction of pullback and homotopy equivalences of relative ∞ -categories.

Proposition B.3. *Consider a diagram $[2] \times [1] \rightarrow \mathrm{Cat}_{\infty}$ depicted as*

$$\begin{array}{ccccc}
& & \mathcal{A} & & \mathcal{B} \\
& \nearrow & \downarrow & \searrow & \\
\mathcal{A}' & \xrightarrow{f'} & \mathcal{B}' & \xrightarrow{p} & \mathcal{C} \\
& \searrow & \nearrow & \downarrow & \\
& & \mathcal{C}' & &
\end{array}$$

$\mathcal{A}' \xrightarrow{f'} \mathcal{B}' \xrightarrow{p} \mathcal{C}'$ $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{q} \mathcal{C}$

$\mathcal{A}' \xrightarrow{f'} \mathcal{B}' \xrightarrow{q'} \mathcal{C}'$ $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{u} \mathcal{C}$

where the squares are all cartesian. Regard \mathcal{A} and \mathcal{B} (resp. \mathcal{A}' and \mathcal{B}') as relative ∞ -categories whose weak equivalences are the maps whose images in \mathcal{C} (resp. \mathcal{C}') are equivalences. Suppose that the following conditions are satisfied:

- (1) The functor $f: \mathcal{A} \rightarrow \mathcal{B}$ is a homotopy equivalence of relative ∞ -categories.
- (2) The functor $u: \mathcal{C}' \rightarrow \mathcal{C}$ is conservative (i.e., reflects equivalences).

Then f' is a homotopy equivalence of relative ∞ -categories.

Proof. We claim that there are maps $g: \mathcal{B} \rightarrow \mathcal{A}$ and $i: \mathcal{B} \rightarrow \mathcal{B}$ in $\mathcal{C}\text{at}_{\infty/\mathcal{C}}$ and a diagram of the form

$$(B.1) \quad \begin{array}{ccc} \mathcal{B} \times \{0\} & \xrightarrow{fg} & \mathcal{B} \\ \downarrow & \nearrow & \downarrow \Phi \\ \mathcal{B} \times \mathcal{I} & \xrightarrow{\Phi} & \mathcal{B} \\ \downarrow & \nearrow & \downarrow \\ \mathcal{B} \times \{1\} & \xrightarrow{i} & \mathcal{B} \end{array}$$

in $\mathcal{C}\text{at}_{\infty/\mathcal{C}}$, where:

- (I) \mathcal{I} is a weakly contractible ∞ -category equipped with two distinguished objects $0, 1 \in \mathcal{I}$.
- (II) For each $B \in \mathcal{B}$, the functor $\Phi|B \times \mathcal{I}$ carries each morphism to a weak equivalence.
- (III) The functor i is an equivalence of ∞ -categories.
- (IV) $\mathcal{B} \times \mathcal{I}$ lies over \mathcal{C} via the composite $\mathcal{B} \times \mathcal{I} \xrightarrow{\text{pr}} \mathcal{B} \xrightarrow{q} \mathcal{C}$.

Changing base along the map $\mathcal{C}' \rightarrow \mathcal{C}$, we obtain functors $g': \mathcal{B}' \rightarrow \mathcal{A}'$, $i': \mathcal{B}' \rightarrow \mathcal{B}'$, and $\Phi': \mathcal{B}' \times \mathcal{I} \rightarrow \mathcal{B}'$. Using condition (2), we find that Φ' witnesses that f' has a right homotopy inverse. Applying the same argument to g , we find that g' also has a right homotopy inverse. It then follows from the two out of six property of homotopy equivalences that f' is a homotopy equivalence of relative categories.

To prove the above claim, use condition (1) to find a relative functor $g: \mathcal{B} \rightarrow \mathcal{A}$ and diagrams of the form (B.1) in $\mathcal{C}\text{at}_\infty$ (not $\mathcal{C}\text{at}_{\infty/\mathcal{C}}$ yet!) that satisfy conditions (I), (II), and (III). Let S denote the set of morphisms of $\mathcal{B} \times \mathcal{I}$ of the form $(B, x) \rightarrow (B, y)$, where $B \in \mathcal{B}$ and $x \rightarrow y$ is a morphism in \mathcal{I} . By condition (I), $q\Phi$ factors through the localization $(\mathcal{B} \times \mathcal{I})[S^{-1}] \simeq \mathcal{B} \times \mathcal{I}[J^{-1}]$. Since \mathcal{I} is weakly contractible, this localization can be identified with \mathcal{B} , with localizing functor given by the projection $\mathcal{B} \times \mathcal{I} \rightarrow \mathcal{B}$. Thus, there is a diagram $[1] \times [1] \rightarrow \mathcal{C}\text{at}_\infty$ of the form

$$\begin{array}{ccc} \mathcal{B} \times \mathcal{I} & \xrightarrow{\Phi} & \mathcal{B} \\ \text{pr} \downarrow & & \downarrow q \\ \mathcal{B} & \xrightarrow{r} & \mathcal{C} \end{array}$$

Since $r \simeq (q\Phi)|\mathcal{B} \times \{1\} = q$, we may assume that $r = q$. Thus we obtain a diagram $\sigma: [2] \rightarrow \mathcal{C}\text{at}_\infty$ whose boundary is depicted as

$$\begin{array}{ccc} \mathcal{B} \times \mathcal{I} & \xrightarrow{\Phi} & \mathcal{B} \\ & \searrow q \circ \text{pr} & \downarrow q \\ & & \mathcal{C}. \end{array}$$

We now contemplate the diagrams $([1] \times [1])^\triangleright \rightarrow \mathbf{Cat}_\infty$ and $[3] \rightarrow \mathbf{Cat}_\infty$ depicted as

The left-hand diagram is constructed as follows: Think of this as an amalgamation of two 3-simplices $\Delta^3 \rightarrow \mathbf{Cat}_\infty$. The front 3-simplex is obtained by filling the horn $\Lambda_1^3 \subset \Delta^3$, using σ , the natural equivalence $fg \simeq \Phi|B \times \{0\}$, and the 2-simplex corresponding to the equality $q \circ \text{pr}|B \times \{0\} = q$. We then fill the 3-simplex in the back by filling the horn $\Lambda_2^3 \subset \Delta^3$. Likewise, the right-hand 3-simplex is obtained by filling the inner horn $\Lambda_1^3 \subset \Delta^3$. These diagrams lifts the maps g, i , and Φ to those in $\mathbf{Cat}_{\infty/C}$, and they give rise to a diagram in $\mathbf{Cat}_{\infty/C}$ satisfying conditions (I) through (IV). The proof is now complete. \square

B.2. Bousfield localization. In this subsection, we record a proof of the following well-known result on Bousfield localization. It has been certainly floating in the literature for quite some time, but we could not find a proof.

Proposition B.4. *Let \mathbf{M} be a combinatorial left proper model category with underlying ∞ -category \mathcal{M} , and let $F: \mathcal{M} \rightarrow \mathcal{N}$ be a functor admitting a fully faithful right adjoint, where \mathcal{N} is presentable. There is a left Bousfield localization \mathbf{N} of \mathbf{M} whose weak equivalences are the maps inverted by the composite $\mathbf{M} \rightarrow \mathcal{M} \xrightarrow{F} \mathcal{N}$. Moreover, the induced functor $\mathbf{N} \rightarrow \mathcal{N}$ is a localization at weak equivalences.*

Proof. Recall from [Lur09, Proposition 5.5.4.2] that there is a small set S of morphisms of \mathcal{M} with the following property: An object $M \in \mathcal{M}$ lies in the essential image of the right adjoint $G: \mathcal{N} \rightarrow \mathcal{M}$ if and only if it is **S -local**, i.e., $\mathcal{M}(f, M)$ is an equivalence for every $f \in S$. By [Bar10, Theorem 4.7], there is a left Bousfield localization $L_S \mathbf{M}$ of \mathbf{M} at S , which has the following description:

- Fibrant objects are the fibrant objects of \mathbf{M} whose image in \mathcal{M} is S -local.
- Weak equivalences are the maps inverted by the composite $\mathbf{M} \rightarrow \mathcal{M} \xrightarrow{F} \mathcal{N}$.

Using the universal property of localization, we obtain functors $\phi: \mathcal{M} \rightarrow (L_S \mathbf{M})[\text{weq}^{-1}]$ and $\psi: (L_S \mathbf{M})[\text{weq}^{-1}] \rightarrow \mathcal{N}$, and the diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{F} & \mathcal{N} \\ & \phi \searrow & \swarrow \psi \\ & (L_S \mathbf{M})[\text{weq}^{-1}] & \end{array}$$

commutes up to natural equivalence. The functors F and ϕ are localizations because it has a fully faithful right adjoint [Lur25, 04JL]. Moreover, a morphism of \mathcal{M} is inverted by F if and only if it is inverted by ϕ . This means that F and ϕ are localization at the same class of maps, so ψ is an equivalence. It follows that the functor $L_S \mathbf{M} \rightarrow \mathcal{N}$ is a localization. Therefore, the left Bousfield localization $L_S \mathbf{M}$ has the desired properties. \square

Remark B.5. An analog of Proposition B.4 is true in the semi-model categorical setting. More precisely, if we assume that \mathbf{M} is a tractable semi-model category, then it admits a (semi-model categorical) left Bousfield localization \mathbf{N} satisfying the conditions in Proposition B.4. The proof is essentially the same, using [BW24, Theorem A] in place of [Bar10, Theorem 4.7].

APPENDIX C. ENRICHED MURO'S THEOREM

In this section, we show that monoidal \mathbf{V} -model categories can be made into ones with a cofibrant unit by changing the class of cofibrations. The result is a minor generalization of [Mur15, Theorem 1], which establishes the corresponding theorem in the unenriched setting.

Proposition C.1. *Let \mathbf{V} be a combinatorial symmetric monoidal model category with a cofibrant unit, and let \mathbf{M} be a combinatorial monoidal \mathbf{V} -model category. There is another combinatorial monoidal \mathbf{V} -model structure on the underlying \mathbf{V} -monoidal category of \mathbf{M} , denoted by $\widetilde{\mathbf{M}}$, satisfying the following conditions:*

- (1) *The weak equivalences of $\widetilde{\mathbf{M}}$ are the weak equivalences of \mathbf{M} .*
- (2) *Every cofibration of \mathbf{M} is a cofibration of $\widetilde{\mathbf{M}}$.*
- (3) *The unit object is cofibrant in $\widetilde{\mathbf{M}}$.*

Moreover, it satisfies the following universal property:

- (4) *Let \mathbf{N} be another monoidal partial \mathbf{V} -model category with a cofibrant unit. Then every monoidal left Quillen \mathbf{V} -functor $\mathbf{M} \rightarrow \mathbf{N}$ extends uniquely to a left Quillen \mathbf{V} -functor $\widetilde{\mathbf{M}} \rightarrow \mathbf{N}$.*

An analogous claim holds for tractable semi-model categories.

Proof. We consider the case of model categories. We will follow [Mur15]. We will use the following notation:

- Given a morphism $f: A \rightarrow A'$ in \mathbf{V} or \mathbf{M} and a morphism $g: B \rightarrow B'$ in \mathbf{V} or \mathbf{M} , we write $f \square g$ for their **pushout product**

$$A \otimes B' \amalg_{A \otimes B} A' \otimes B \rightarrow A' \otimes B'$$

whenever the notation makes sense (i.e., either (i) both f and g are maps of \mathbf{V} , (ii) f is a morphism of \mathbf{V} and g is a morphism of \mathbf{M} , or (iii) both f and g are morphisms of \mathbf{M} .)

- If S and T are collections of morphisms in \mathbf{V} or \mathbf{M} , we will write $S \square T = \{s \square t \mid s \in S, t \in T\}$ whenever the right hand side makes sense.
- If $i: A \rightarrow B$ and $p: X \rightarrow Y$ are morphisms of \mathbf{M} , we will write $\langle i, p \rangle$ for their **pullback power**

$$\mathbf{M}(B, X) \rightarrow \mathbf{M}(A, X) \times_{\mathbf{M}(A, Y)} \mathbf{M}(B, Y),$$

where we wrote $\mathbf{M}(-, -)$ for the enriched hom-objects of \mathbf{M} .

- The symbol $\mathbf{1}$ will always mean the unit of \mathbf{M} ; when we need to refer to the unit of \mathbf{V} (which happens only once in the proof), we denote it by $\mathbf{1}_{\mathbf{V}}$.

Choose a weak equivalence $q: \widetilde{\mathbf{1}} \rightarrow \mathbf{1}$ which witnesses Muro's unit axiom in \mathbf{M} . Choose a factorization

$$\widetilde{\mathbf{1}} \amalg \mathbf{1} \xrightarrow{\iota} C \xrightarrow{\pi} \mathbf{1}$$

of the map (q, id) , where ι is a cofibration in \mathbf{M} and π is a weak equivalence in \mathbf{M} . Let $\text{inc}_1: \widetilde{\mathbf{1}} \rightarrow \widetilde{\mathbf{1}} \amalg \mathbf{1}$ denote the inclusion of the first summand, and set $\kappa = \iota \circ \text{inc}_1: \widetilde{\mathbf{1}} \rightarrow C$. Also, choose generating sets $I_{\mathbf{M}}$ and $J_{\mathbf{M}}$ of cofibrations and trivial cofibrations of \mathbf{M} , and choose generating sets $I_{\mathbf{V}}$ and $J_{\mathbf{V}}$ of cofibrations and trivial cofibrations of \mathbf{V} . By enlarging $I_{\mathbf{V}}$ if necessary, we may assume that the map $\emptyset \rightarrow \mathbf{1}_{\mathbf{V}}$ belongs to $I_{\mathbf{V}}$. Let $\phi: \emptyset \rightarrow \mathbf{1}$ denote the unique map from the initial object. We set

$$\begin{aligned} I_{\widetilde{\mathbf{M}}} &= I_{\mathbf{M}} \cup (I_{\mathbf{V}} \square \{\phi\}), \\ J_{\widetilde{\mathbf{M}}} &= J_{\mathbf{M}} \cup (I_{\mathbf{V}} \square \{\kappa\}) \cup (J_{\mathbf{V}} \square \{\phi\}). \end{aligned}$$

We claim that there is a cofibrantly generated model structure on \mathbf{M} with generating sets of cofibrations and trivial cofibrations given by $I_{\widetilde{\mathbf{M}}}$ and $J_{\widetilde{\mathbf{M}}}$, and with weak

equivalences given by those of \mathbf{M} . Write $I_{\widetilde{\mathbf{M}}}\text{-cof}$ and $I_{\widetilde{\mathbf{M}}}\text{-inj}$ for the classes of $I_{\widetilde{\mathbf{M}}}$ -cofibrations and $I_{\widetilde{\mathbf{M}}}$ -injectives; we also make use of similar notations, such as $J_{\mathbf{V}}\text{-cof}$ and $J_{\widetilde{\mathbf{M}}}\text{-inj}$, and let W denote the class of weak equivalences of \mathbf{M} . According to [Hir03, Theorem 11.3.1], we must verify the following:

- (a) $J_{\widetilde{\mathbf{M}}}\text{-cof} \subset (I_{\widetilde{\mathbf{M}}}\text{-cof}) \cap W$. (In the case of semi-model categories, we replace $J_{\widetilde{\mathbf{M}}}\text{-cof}$ by its subclass of maps with cofibrant domain.)
- (b) $I_{\widetilde{\mathbf{M}}}\text{-inj} \subset (J_{\widetilde{\mathbf{M}}}\text{-inj}) \cap W$.
- (c) $(J_{\widetilde{\mathbf{M}}}\text{-inj}) \cap W \subset I_{\widetilde{\mathbf{M}}}\text{-inj}$.

Before we prove (a) through (c), we observe that the associativity (up to natural isomorphism) of pushout-product implies the following inclusion:

- (d) $(I_{\mathbf{V}}\text{-cof}) \square (I_{\widetilde{\mathbf{M}}}\text{-cof}) \subset (I_{\widetilde{\mathbf{M}}}\text{-cof})$
- (e) $((J_{\mathbf{V}}\text{-cof}) \square (I_{\widetilde{\mathbf{M}}}\text{-cof})) \subset J_{\widetilde{\mathbf{M}}}\text{-cof}$.
- (f) $((I_{\mathbf{V}}\text{-cof}) \square (J_{\widetilde{\mathbf{M}}}\text{-cof})) \subset J_{\widetilde{\mathbf{M}}}\text{-cof}$.

Note that (d) implies:

- (d') A map p of \mathbf{M} is an $I_{\widetilde{\mathbf{M}}}$ -injective if and only if, for each $I_{\widetilde{\mathbf{M}}}$ -cofibration i , the map $\langle i, p \rangle$ is a trivial fibration of \mathbf{V} .

(There is a similar characterization of $J_{\widetilde{\mathbf{M}}}$ -injectives, but we will not need it in the sequel.)

We also observe that:

- (*) The functor $-\otimes\widetilde{\mathbf{1}}:\mathbf{M}\rightarrow\mathbf{M}$ preserves and detects weak equivalences.
- (**) The morphism $\kappa\otimes\widetilde{\mathbf{1}}$ is a $J_{\mathbf{M}}$ -cofibration.

Indeed, assertion (*) is clear because $-\otimes\widetilde{\mathbf{1}}$ is naturally weakly equivalent to the identity functor. Assertion (**) is a consequence of the fact that $\kappa\otimes\widetilde{\mathbf{1}}$ is a $I_{\mathbf{M}}$ -cofibration (being the composite of a pushout of $\emptyset\rightarrow\widetilde{\mathbf{1}}$ and the map $\iota\otimes\widetilde{\mathbf{1}}$) and is a weak equivalence by (*).

Now we prove (a) through (c). We start from (a). To show that $J_{\widetilde{\mathbf{M}}}\text{-cof} \subset I_{\widetilde{\mathbf{M}}}\text{-cof}$, it will suffice to show that $J_{\widetilde{\mathbf{M}}} \subset I_{\widetilde{\mathbf{M}}}\text{-cof}$. For this, in light of (d), it suffices to show that κ is an $I_{\widetilde{\mathbf{M}}}$ -cofibration. This is clear, because it is the composite of the pushout of ϕ and the $I_{\mathbf{M}}$ -cofibration ι . To show that $J_{\widetilde{\mathbf{M}}}\text{-cof} \subset W$, we observe that $(J_{\widetilde{\mathbf{M}}}\text{-cof}) \otimes \widetilde{\mathbf{1}} \subset J_{\mathbf{M}}\text{-cof}$ by (**). It follows from (*) that $J_{\widetilde{\mathbf{M}}}\text{-cof} \subset W$, as desired.

Next, we prove (b). Let p be an $I_{\widetilde{\mathbf{M}}}$ -injective. We must show that $p \in (J_{\widetilde{\mathbf{M}}}\text{-inj}) \cap W$. Since $I_{\widetilde{\mathbf{M}}}\text{-inj} \subset I_{\mathbf{M}}\text{-inj} = (J_{\mathbf{M}}\text{-inj}) \cap W$, it suffices to show that $\langle \kappa, p \rangle$ is a trivial fibration and $\langle \phi, p \rangle$ is a fibration. By (d'), this will follow if κ and ϕ are $I_{\widetilde{\mathbf{M}}}$ -cofibrations. For ϕ , this is clear. We have also seen in the previous paragraph that κ is an $I_{\widetilde{\mathbf{M}}}$ -cofibration. Thus we have proved (b).

Finally, we prove (c). Let $p \in (J_{\widetilde{\mathbf{M}}}\text{-inj}) \cap W$. We wish to show that $p \in I_{\widetilde{\mathbf{M}}}\text{-inj}$. Since $J_{\widetilde{\mathbf{M}}}\text{-inj} \subset J_{\mathbf{M}}\text{-inj} = (I_{\mathbf{M}}\text{-inj}) \cap W$, it suffices to show that $\langle \phi, p \rangle$ is a trivial fibration. Since ϕ is a retract of the map $\psi:\emptyset\rightarrow C$, it suffices to show that $\langle \psi, p \rangle$ is a trivial fibration. For this, factor the map ψ as

$$\emptyset \xrightarrow{\widetilde{\phi}} \widetilde{\mathbf{1}} \xrightarrow{\kappa} C.$$

The map $\widetilde{\phi}$ is an $I_{\mathbf{M}}$ -cofibration, so $\langle \widetilde{\phi}, p \rangle$ is a trivial fibration. The map $\langle \kappa, p \rangle$ is a trivial fibration by the definition of $J_{\widetilde{\mathbf{M}}}$ -injectives. Since $\langle \psi, p \rangle$ is the composition of $\langle \kappa, p \rangle$ and a pullback of $\langle \widetilde{\phi}, p \rangle$, this proves that $\langle \psi, p \rangle$ is a trivial fibration, as desired.

We have thus shown that the sets $I_{\widetilde{\mathbf{M}}}$, $J_{\widetilde{\mathbf{M}}}$ and the class of weak equivalences of \mathbf{M} gives rise to a model structure on \mathbf{M} , which we denote by $\widetilde{\mathbf{M}}$. We complete the proof by showing that $\widetilde{\mathbf{M}}$ satisfies conditions (1) through (5). Conditions (1) through (4) are immediate from the constructions, so we focus on (5).

Suppose that \mathbf{M} is a monoidal model category. We must show that so is $\widetilde{\mathbf{M}}$, i.e., that $I_{\widetilde{\mathbf{M}}} \square I_{\widetilde{\mathbf{M}}} \subset I_{\widetilde{\mathbf{M}}}\text{-cof}$, $I_{\widetilde{\mathbf{M}}} \square J_{\widetilde{\mathbf{M}}} \subset J_{\widetilde{\mathbf{M}}}\text{-cof}$, and $J_{\widetilde{\mathbf{M}}} \square I_{\widetilde{\mathbf{M}}} \subset J_{\widetilde{\mathbf{M}}}\text{-cof}$. Using the fact that $\widetilde{\mathbf{M}}$ is a monoidal \mathbf{V} -model category, and also using inclusions (a), (d), and (e), we are reduced to showing the following:

- (****) For every $I_{\mathbf{M}}$ -cofibration i , the maps $i \square \kappa$ and $\kappa \square i$ are $J_{\mathbf{M}}$ -cofibrations (and hence $J_{\widetilde{\mathbf{M}}}$ -cofibrations).

To prove (***), we observe that $i \square \kappa$ is a composition of $i \square \iota$ and a pushout of $i \otimes \phi \cong i$. Since the maps $i \square \iota$ and i are $I_{\mathbf{M}}$ -cofibrations, we deduce that $i \square \kappa$ is an $I_{\mathbf{M}}$ -cofibration. Moreover, we know from (**) that $\kappa \otimes \widetilde{\mathbf{1}}$ is an $J_{\mathbf{M}}$ -cofibration, so $(i \square \kappa) \otimes \widetilde{\mathbf{1}}$ is also a $J_{\mathbf{M}}$ -cofibration. It follows from (*) that $i \square \kappa$ is a weak equivalence. Hence $i \square \kappa$ is a $J_{\mathbf{M}}$ -cofibration. Likewise, $\kappa \square i$ is a $J_{\mathbf{M}}$ -cofibration. The proof is now complete. \square

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