ON THE EQUIVALENCE OF BRANTNER AND CHU–HAUGSENG'S APPROACHES TO ENRICHED ∞-OPERADS

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Abstract. We prove that two models of (monochromatic) ∞-operads, due to Brantner and Chu–Haugseng, are equivalent. The equivalence will be proved as a consequence of the equivalence of two models of monoidal ∞-categories of symmetric sequences and composition product, due to Brantner and Haugseng.

Introduction

Operads were introduced by Boardman–Vogt [BV73] and May [May72] as a tool to describe algebraic structures inherent in topological spaces. Soon after the introduction, Kelly [Kel05] introduced a generalized notion of *enriched operads* to describe algebraic structures of objects in symmetric monoidal categories. Since their introduction, enriched operads have become indispensable in a wide range of areas, from algebraic topology, algebraic geometry, homological algebra, mathematical physics, and computer science to geometric topology [May72, BV73, Val14, LV12, BF21, BdBW13].

Despite its successes, the classical theory of enriched operads exhibits limitations in homotopical settings. In particular, when one wishes to treat enriched operads and their algebras "up to homotopy," the classical theory turns out to be too rigid. For example, in many cases, meaningful homotopy theory of algebras is available only under strong cofibrancy assumptions on the operad (such as Σ-cofibrancy). This motivates the study of *enriched ∞-operads*, which describe algebraic structures in symmetric monoidal ∞-categories and provide a more flexible framework.

As with many higher categorical generalizations of classical notions, enriched ∞-operads admit multiple constructions. One approach was developed in the work of Chu–Haugseng [CH20], where the authors gave several equivalent definitions of enriched ∞-operads using "Segal-type" conditions. Around the same time, Brantner constructed a different model of enriched ∞-operads [Bra17].

	Developing abstract theories	Applications
Chu–Haugseng	Yes	No
Brantner	No	Yes

Table 1. A quick comparison

These approaches have complementary strengths. On the one hand, Chu–Haugseng's approach is well-suited for developing abstract theories of enriched ∞-operads, and a substantial body of such theory has already been developed with their approach. For example, it is compatible with the classical theory of ∞-operads enriched in spaces, or more generally, nice model categories [CH20, Theorem 5.2.10]; algebras over enriched ∞-operads in this sense are also known to be well-behaved and are compatible with their strict counterpart [Hau19, Theorem 4.10]; and they admit a characterization in terms of maps into endomorphism ∞-operads [Hau19]. However,

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their approach is rather heavy and does not seem to be used much in applications. In contrast, Brantner's approach is comparatively lightweight and has been used extensively in the literature [FG12, Knu18, Ama22, BCN23, Heu24, ACBH25, HL24]. However, it is somewhat awkward to develop abstract theories of enriched ∞-operads with his approach. For example, it is not apparent from the definition that his construction is functorial in the enriching symmetric monoidal ∞-categories! The contrast is summarized in Table 1.

Given the complementary roles of the two approaches, it is desirable to establish an equivalence of these models. However, such a comparison has remained unavailable, resulting in two possibly different definitions of enriched ∞-operads. The goal of this paper is to resolve this issue definitively.

To explain our result, we recall a standard perspective on operads. Namely, monochromatic operads in a symmetric monoidal category can be described as an algebra object in the monoidal category of *composition product* of *symmetric sequences.* Brantner and Haugseng constructed two seemingly different ∞-categorical refinements of the composition product monoidal structure [Bra17, Hau22]. More precisely, to each presentably symmetric monoidal ∞-category $\mathcal{C}$, they associated monoidal ∞-categories $\Sigma\mathrm{Seq}_{\mathrm{B}}(\mathcal{C})$ and $\Sigma\mathrm{Seq}_{\mathrm{H}}(\mathcal{C})$. The algebra objects in these monoidal ∞-categories recover Brantner's and Chu–Haugseng's enriched ∞-operads, respectively. Our main theorem then asserts that:

Theorem A (Theorem 1.17)**.** There is a natural equivalence of monoidal ∞-categories

$$\Sigma\mathrm{Seq}_{\mathrm{B}}(\mathcal{C}) \simeq \Sigma\mathrm{Seq}_{\mathrm{H}}(\mathcal{C}).$$

As an immediate consequence, we find that Brantner and Chu–Haugseng's definitions of monochromatic enriched ∞-operads are equivalent. Moreover, Theorem A gives something stronger. Indeed, the monoidal equivalence ensures that two definitions of Koszul duality of enriched ∞-operads, based on the two models of enriched ∞-operads, are equivalent to each other. To our knowledge, this comparison has not previously appeared in the literature.

Our strategy for the proof of Theorem A is relatively simple: We "de-localize" each monoidal ∞-category to a model category. That this will produce a natural equivalence is ensured by the main theorem of [Arab], which asserts that the homotopy theory of (combinatorial) symmetric monoidal model categories is equivalent to that of presentably symmetric monoidal ∞-categories. At present, we do not know a direct ∞-categorical proof of Theorem A. The absence of such a proof reflects the genuine subtlety of the problem rather than a deficiency of the approach. While the underlying idea is simple, its implementation requires substantial technical work, as reflected in the length of the paper.

Organization of the paper. This paper consists of 6 sections and an appendix.

- In Section 1, we review the definitions of Brantner and Haugseng's composition product monoidal structure, and then state our main theorem precisely.
- In Section 2, we recall the definition of monoidal localization and a few related results.
- In Section 3, we establish a few key results on Day's convolution product and composition product from a model-categorical perspective.
- In Sections 4 and 5, we show that both Brantner and Haugseng's models are characterized by certain universal property involving symmetric monoidal model categories and their localization.
- In Section 6, we give the proof of the main theorem.
- Appendix A is a brief summary of results on ∞-bicategories we need in the main body.

Notation and convention.

- Throughout the paper, we use the word **∞-category** as a synonym for quasicategory in the sense of [Joy02]. We will mainly follow [Lur09b] and [Lur17] in various terminology and notation related to ∞-categories.
- We will generally not notationally distinguish between categories and their nerves, and bicategories with their Duskin nerves (Example A.9).
- If $\mathcal{C}$ is an ∞-category, we denote its maximal sub Kan complex by $\mathcal{C}^{\simeq}$ and refer to it as the **core** of $\mathcal{C}$.
- We write Fin for the skeleton of the category of finite sets and set maps. Explicitly, its objects are the sets $\underline{n} = \{1, \dots, n\}$ for $n \geq 0$. Various set-theoretical operations will be replaced by categorical operations in this category. For example, if $f \colon S \to T$ is a map in Fin and $t \in T$, then we write $f^{-1}(t) \in \mathsf{Fin}$ for the category-theoretic (not set-theoretic) fiber of f over t.
- We write Fin_* for the skeleton of the category of finite pointed sets and set maps. Explicitly, its objects are $\langle n \rangle = (\{*, 1, \dots, n\}, *)$ for $n \geq 0$.
- We write FB for the maximal subgroupoid of Fin.
- Following [Lur17], we use symbols like $\mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ to denote a symmetric monoidal ∞-category with underlying ∞-category $\mathcal{C}$. We typically denote the unit object of $\mathcal{C}^{\otimes}$ by $\mathbf{1}_{\mathcal{C}} = \mathbf{1}$. We typically implicitly identify Fin_* with the larger category of all finite pointed sets and pointed maps by choosing an inverse equivalence.
- If $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ are symmetric monoidal ∞-categories, we write $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ for the ∞-category of symmetric monoidal functors $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ [Lur17, Definition 2.1.3.7]. Similar notation will be used for monoidal ∞-categories.
- We write $\mathcal{M}\mathsf{on}\mathcal{C}\mathsf{at}_{\infty}$ for the ∞-category of small monoidal ∞-categories. Formally, it is defined as the homotopy coherent nerve of the simplicial category whose objects are the small monoidal ∞-categories, and whose mapping simplicial sets are given by $\mathrm{Fun}^{\otimes}(-,-)^{\simeq}$. (Equivalently, it is the ∞-categorical localization of the ordinary category of monoidal ∞-categories and monoidal functors at equivalences of monoidal ∞-categories.) We define the ∞-category of *large* monoidal ∞-categories $\mathcal{M}\mathsf{on}\widehat{\mathcal{C}\mathsf{at}}_{\infty}$ similarly.
- If $(\mathcal{M}, \otimes, \mathbf{1})$ is a symmetric monoidal category in the ordinary sense, we write $\mathcal{M}^{\otimes} \to \mathsf{Fin}_*$ for the associated symmetric monoidal ∞-category [Lur17, Construction 2.0.0.1].

1. Stating the main result

Let $\mathcal{C}$ be a cocomplete closed symmetric monoidal category. A **symmetric sequence** in $\mathcal{C}$ is a functor $X \colon \mathsf{FB} \to \mathcal{C}$. Such a functor consists of a sequence $(X(0), X(1), \dots)$ of objects in $\mathcal{C}$, where each $X(n)$ carries a Σ_n-action, hence the name. Given two symmetric sequences X, Y in $\mathcal{C}$, their **composition product** $X \circ Y$ is defined by the formula[1]

$$X \circ Y(A) = \mathrm{colim}_{f \in A \to B \in \mathsf{FB} \times_{\mathsf{Fin}} \mathsf{Fin}_{A/}} \left(\bigotimes_{b \in B} X\left(f^{-1}(b)\right) \right) \otimes Y(B).$$

The category $\Sigma\mathrm{Seq}(\mathcal{C}) = \mathrm{Fun}(\mathsf{FB}, \mathcal{C})$ of symmetric sequences is a monoidal category with respect to $\circ$ and the unit symmetric sequence $(\emptyset, \mathbf{1}, \emptyset, \emptyset, \dots)$. Monoid objects in this monoidal category are exactly (monochromatic) operads enriched in $\mathcal{C}$.

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[1] In many classical literature, the composition product is written in the reverse order. In other words, what we just defined as $X \circ Y$ is often denoted by $Y \circ X$. We follow [Hau22] in our convention here.

In this section, we review the ∞-categorical enhancements of the composition product monoidal structure, due to Brantner and Chu–Haugseng. We then state the main result of this paper, which compares the two monoidal structures.

1.1. **Brantner's model.**

Branter's model of composition product is based on the following observation, found in Trimble's note [Tri] and "Author's Note" of [Kel05] (there attributed to Carboni): The groupoid FB carries a symmetric monoidal structure, given by disjoint union of sets. This (and its opposite) is the free symmetric monoidal category on a singleton $\underline{1}$. It follows that the symmetric monoidal category $\mathrm{Fun}\left(\mathsf{FB},\mathsf{Set}\right)$ of symmetric sequences with the and Day's convolution product $\bigstar$ is the free cocomplete closed symmetric monoidal category generated by the unit symmetric sequence $\mathfrak{X}=\mathsf{FB}\left(\underline{1},-\right)$ [IK86, Theorem 5.1]. In analogy with ordinary algebra, let us therefore write $\mathsf{Set}[\mathfrak{X}]=\mathrm{Fun}\left(\mathsf{FB},\mathsf{Set}\right)$ for this symmetric monoidal category. The universal property implies that the category $\mathrm{Fun}^{\otimes,L}\left(\mathsf{Set}[\mathfrak{X}],\mathsf{Set}[\mathfrak{X}]\right)$ of cocontinuous symmetric monoidal functors $\mathsf{Set}[\mathfrak{X}]\to\mathsf{Set}[\mathfrak{X}]$ is equivalent $\mathsf{Set}[\mathfrak{X}]=\Sigma\mathrm{Seq}\left(\mathsf{Set}\right)$, via evaluation at $\mathfrak{X}\in\mathsf{Set}[\mathfrak{X}]$. We then have:

Proposition 1.1. *The categorical equivalence*

$$\mathrm{ev}_{\mathfrak{X}}\colon\ \mathrm{Fun}^{\otimes,L}\left(\mathsf{Set}[\mathfrak{X}],\mathsf{Set}[\mathfrak{X}]\right)\xrightarrow{\simeq}\Sigma\mathrm{Seq}\left(\mathsf{Set}\right)$$

can be enhanced to a monoidal equivalence, where the left-hand carries the monoidal structure given by composition of symmetric monoidal functors, and the right-hand side carries the composition product monoidal structure.

Proof. Since the Yoneda embedding $\mathsf{FB}^{\mathrm{op}}\to\mathsf{Set}[\mathfrak{X}]$ is symmetric monoidal, we have $\mathsf{FB}\left(S,-\right)\cong\mathfrak{X}^{\bigstar S}$ for every finite set $S\in\mathsf{FB}$. Therefore, the co-Yoneda lemma gives us an isomorphism

$$F\cong\int^{S\in\mathsf{FB}}\mathfrak{X}^{\bigstar S}\cdot F\left(S\right)$$

natural in $F\in\mathsf{Set}[\mathfrak{X}]$. (A more suggestive notation for the right-hand side will be $\sum_{n\geq0}F\left(\underline{n}\right)\times_{\Sigma_n}\mathfrak{X}^{\boxtimes n}$.) Thus, if $\overline{F},\overline{G}\colon\mathsf{Set}[\mathfrak{X}]\to\mathsf{Set}[\mathfrak{X}]$ are objects of $\mathrm{Fun}^{\otimes,L}\left(\mathsf{Set}[\mathfrak{X}],\mathsf{Set}[\mathfrak{X}]\right)$ with images $F,G\in\Sigma\mathrm{Seq}\left(\mathsf{Set}\right)$, then we have

$$\begin{aligned}\mathrm{ev}_{\mathfrak{X}}\left(\overline{F}\circ\overline{G}\right)&=\overline{F}\left(G\right)\\&\cong\overline{F}\left(\int^{S\in\mathsf{FB}}\mathfrak{X}^{\bigstar S}\cdot G\left(S\right)\right)\\&\cong\int^{S\in\mathsf{FB}}F^{\bigstar S}\cdot G\left(S\right)\\&\cong F\circ G.\end{aligned}$$

These isomorphisms and the identity morphism $\mathrm{ev}_{\mathfrak{X}}\left(\mathrm{id}_{\mathsf{Set}[\mathfrak{X}]}\right)=\mathfrak{X}$ enhances $\mathrm{ev}_{\mathfrak{X}}$ to a monoidal functor. □

The goal of this subsection is to describe the ∞-categorical generalization of this story, due to Brantner.

Remark 1.2. As is clear from the above discussion, Brantner's model lives in the world of $(\infty,2)$-categories. In what follows, we will use ∞-bicategories as our preferred model of $(\infty,2)$-categories. A brief summary of this model can be found in Appendix A. We will also make a few reference to Lemma 3.5. No circularity will result from these forward references.

To state the definition of Brantner's composition product monoidal structure, we need to introduce a bit of notation.

Definition 1.3. A **presentably symmetric monoidal ∞-category** is a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ satisfying the following pair of conditions:

- The ∞-category $\mathcal{C}$ is presentable.
- The tensor bifunctor $\mathcal{C}\times\mathcal{C}\to\mathcal{C}$ preserves small colimits in each variable.

Definition 1.4. We define an ∞-bicategory $\mathcal{P}\mathrm{r}\mathcal{SM}_{(2)}$ to be the one associated with (via Recollection A.10) the simplicial category whose:

- objects are the presentably symmetric monoidal ∞-categories; and
- whose hom-simplicial set from $\mathcal{C}^{\otimes}$ to $\mathcal{D}^{\otimes}$ is given by the full subcategory $\mathrm{Fun}^{\otimes,L}\left(\mathcal{C},\mathcal{D}\right)\subset\mathrm{Fun}^{\otimes}\left(\mathcal{C},\mathcal{D}\right)$ spanned by the symmetric monoidal functors whose underlying functor $\mathcal{C}\to\mathcal{D}$ preserves small colimits.

The underlying ∞-category of $\mathcal{P}\mathrm{r}\mathcal{SM}^{(2)}$ will be denoted by $\mathcal{P}\mathrm{r}\mathcal{SM}$.

Notation 1.5. We denote the binary coproduct in $\mathcal{P}\mathrm{r}\mathcal{SM}$ by $\otimes$. More generally, if $\mathcal{B}^{\otimes}\leftarrow\mathcal{A}^{\otimes}\to\mathcal{C}^{\otimes}$ are maps in $\mathcal{P}\mathrm{r}\mathcal{SM}$, we denote the pushout by $\left(\mathcal{B}\otimes_{\mathcal{A}}\mathcal{C}\right)^{\otimes}$. Note that it exists by [Lur17, Corollary 3.2.3.3] and [Lur09b, Theorem 5.5.3.18, Corollary 5.5.3.4, and Remark 5.5.3.9].

Remark 1.6. Pushouts in $\mathcal{P}\mathrm{r}\mathcal{SM}$ satisfies the following $(\infty,2)$-universal property: Given maps $\mathcal{B}^{\otimes}\leftarrow\mathcal{A}^{\otimes}\to\mathcal{C}^{\otimes}$ in $\mathcal{P}\mathrm{r}\mathcal{SM}$ and an object $\mathcal{Z}^{\otimes}\in\mathcal{P}\mathrm{r}\mathcal{SM}$, the functor

$$\theta_{\mathcal{Z}}:\mathrm{Fun}^{\otimes,L}\left(\mathcal{B}\otimes_{\mathcal{A}}\mathcal{C},\mathcal{Z}\right)\to\mathrm{Fun}^{\otimes,L}\left(\mathcal{B},\mathcal{Z}\right)\times_{\mathrm{Fun}^{\otimes,L}\left(\mathcal{A},\mathcal{Z}\right)}\mathrm{Fun}^{\otimes,L}\left(\mathcal{C},\mathcal{Z}\right)$$

is a categorical equivalence. To see this, it suffices to show that for every ∞-category $\mathcal{X}$, the functor $\mathrm{Fun}\left(\mathcal{X},\theta\right)^{\simeq}$ is a homotopy equivalence. But we can identify $\mathrm{Fun}\left(\mathcal{X},\theta\right)^{\simeq}$ with $\left(\theta_{\mathrm{Fun}(\mathcal{X},\mathcal{Z})}\right)^{\simeq}$, which is a homotopy equivalence by the definition of pushouts. (Here $\mathrm{Fun}\left(\mathcal{X},\mathcal{Z}\right)^{\otimes}$ is defined as the fiber product $\mathrm{Fun}\left(\mathcal{X},\mathcal{Z}^{\otimes}\right)\times_{\mathrm{Fun}(\mathcal{X},\mathsf{Fin}_{*})}\mathsf{Fin}_{*}$.)

Notation 1.7. Let $\mathcal{A}^{\otimes}$ be a small symmetric monoidal ∞-category. According to [Lur17, Corollary 4.8.1.12], the ∞-category $\mathrm{Fun}\left(\mathcal{A},\mathcal{S}\right)$ admits a presentably symmetric monoidal structure which is characterized by the property that the Yoneda embedding $\mathcal{A}^{\mathrm{op}}\to\mathrm{Fun}\left(\mathcal{A},\mathcal{S}\right)$ can be enhanced to a symmetric monoidal functor. We let $\mathrm{Fun}\left(\mathcal{A},\mathcal{S}\right)^{\bigstar}$ denote corresponding symmetric monoidal ∞-category.

If $\mathcal{C}^{\otimes}$ is a presentably symmetric monoidal ∞-category, we define another presentably symmetric monoidal ∞-category $\mathrm{Fun}\left(\mathcal{A},\mathcal{C}\right)^{\bigstar}$ and a symmetric monoidal functor $i_{\mathcal{A},\mathcal{C}}$ by the pushout (or coproduct)

$$\begin{tikzcd}
\mathcal{S}^{\times} \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{A},\mathcal{S})^{\bigstar} \arrow[d] \\
\mathcal{C}^{\otimes} \arrow[r, "i_{\mathcal{A},\mathcal{C}}"'] \arrow[ur, phantom, "\ulcorner", very near end] & \mathrm{Fun}(\mathcal{A},\mathcal{C})^{\bigstar}
\end{tikzcd}$$

(Note that the underlying ∞-category of $\mathrm{Fun}\left(\mathcal{A},\mathcal{C}\right)^{\bigstar}$ is equivalent to $\mathrm{Fun}\left(\mathcal{A},\mathcal{C}\right)$ by Lemma 3.5, justifying the notation.)

Notation 1.8. Given a presentably symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$, we write

$$\mathcal{C}[\mathfrak{X}]^{\bigstar}=\mathrm{Fun}\left(\mathsf{FB},\mathcal{C}\right)^{\bigstar},$$

and let $\mathfrak{X}\in\mathcal{C}[\mathfrak{X}]$ denote the image of the unit symmetric sequence $\mathfrak{X}=\left(\emptyset,\underline{1},\emptyset,\emptyset,\dots\right)\in\mathrm{Fun}\left(\mathsf{FB},\mathcal{S}\right)$. We also write $i_{\mathcal{C}}=i_{\mathsf{FB}^{\mathrm{op}},\mathcal{C}}$ for the symmetric monoidal functor $\mathcal{C}^{\otimes}\to\mathcal{C}[\mathfrak{X}]^{\bigstar}$.

Remark 1.9. In the situation of Notation 1.8, Lemma 3.5 shows that an object of $\mathcal{C}[\mathfrak{X}]$ can informally be written as a sequence $\left(X\left(0\right),X\left(1\right),\dots\right)$ of objects in $\mathcal{C}$, where each $X(n)$ carries a Σ_n-action. The functor $i_{\mathcal{C}}$ of Notation 1.8 is then given by the formula

$$i_{\mathcal{C}}(C) = (C, \emptyset, \emptyset, \dots).$$

We can now define Brantner's composition product monoidal structure.

Definition 1.10. Let $\mathcal{C}^{\otimes}$ be a presentably symmetric monoidal ∞-category. We define a monoidal ∞-category $\Sigma\mathrm{Seq}_{\mathrm{B}}(\mathcal{C})^{\circ}$ as the endomorphism monoidal ∞-category (Definition A.41) of the ∞-bicategory $\mathrm{CAlg}^{(2)}_{\mathcal{C}} = \left(\mathcal{P}\mathsf{r}\mathcal{SM}^{(2)}\right)^{\mathcal{C}^{\otimes}/}$ (Example A.4) at $\mathcal{C}[\mathfrak{X}]^{\bigstar}$. In symbols, we have

$$\Sigma\mathrm{Seq}_{\mathrm{B}}(\mathcal{C})^{\circ} = \mathcal{E}\mathrm{nd}_{\mathrm{CAlg}^{(2)}_{\mathcal{C}}}\left(\mathcal{C}[\mathfrak{X}]^{\bigstar}\right)^{\circ}.$$

Remark 1.11. The assignment $\mathcal{C}^{\otimes} \mapsto \Sigma\mathrm{Seq}_{\mathrm{B}}(\mathcal{C})^{\circ}$ can be assembled into a functor

$$\Sigma\mathrm{Seq}_{\mathrm{B}}(-)^{\circ} : \mathcal{P}\mathsf{r}\mathcal{SM} \to \mathcal{M}\mathsf{on}\widehat{\mathcal{C}\mathsf{at}}_{\infty}$$

as follows: Define an ∞-bicategory $\int^{\mathcal{P}\mathsf{r}\mathcal{SM}} \mathrm{CAlg}^{(2)}_{\bullet}$ by the pullback

$$\begin{tikzcd}
\int^{\mathcal{P}\mathsf{r}\mathcal{SM}} \mathrm{CAlg}^{(2)}_{\bullet} \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Fun}^{\mathrm{bi}}([1], \mathcal{P}\mathsf{r}\mathcal{SM}^{(2)}) \arrow[d, "\mathrm{ev}_0"] \\
\mathcal{P}\mathsf{r}\mathcal{SM} \arrow[r] & \mathcal{P}\mathsf{r}\mathcal{SM}^{(2)}.
\end{tikzcd}$$

By Example A.20 and Remark 1.6, the functor π is a cocartesian fibration (Definition A.17). We let $\mathrm{CAlg}^{(2)}_{\bullet} \colon \mathcal{P}\mathsf{r}\mathcal{SM} \to \mathcal{B}\mathsf{i}\mathcal{C}\mathsf{at}_{\infty}$ denote the straightening (Theorem A.22) of π.

Since $\mathcal{S}^{\times} \in \mathcal{P}\mathsf{r}\mathcal{SM}$ is initial, there is an (essentially unique) cocartesian section σ of π which carries $\mathcal{S}^{\times}$ to the inclusion

$$i_{\mathcal{S}} \colon \mathcal{S}^{\times} \hookrightarrow \mathcal{S}[\mathfrak{X}]^{\bigstar}.$$

Via straightening, the section σ lifts the functor $\mathrm{CAlg}^{(2)}_{\bullet}$ to a functor

$$\mathcal{P}\mathsf{r}\mathcal{SM} \to (\mathcal{B}\mathsf{i}\mathcal{C}\mathsf{at}_{\infty})_{[0]/}.$$

Composing this with the functor $\mathcal{E}\mathrm{nd} \colon (\mathcal{B}\mathsf{i}\mathcal{C}\mathsf{at}_{\infty})_{[0]/} \to \mathcal{M}\mathsf{on}\mathcal{C}\mathsf{at}_{\infty}$ of Definition A.41, we get the desired functor

$$\Sigma\mathrm{Seq}_{\mathrm{B}}(-)^{\circ} : \mathcal{P}\mathsf{r}\mathcal{SM} \to \mathcal{M}\mathsf{on}\mathcal{C}\mathsf{at}_{\infty}.$$

The notation $\Sigma\mathrm{Seq}_{\mathrm{B}}(\mathcal{C})$ is justified by the following proposition:

Proposition 1.12. *Let $\mathcal{C}^{\otimes}$ be a presentably symmetric monoidal ∞-category. Then $\mathcal{C}[\mathfrak{X}]^{\bigstar} \in \mathrm{CAlg}^{(2)}_{\mathcal{C}}$ is freely generated by $\mathfrak{X}$ in the following sense: For every $\mathcal{D}^{\otimes} \in \mathrm{CAlg}^{(2)}_{\mathcal{C}}$, the evaluation at $\mathfrak{X}$ induces a categorical equivalence*[2]

$$\mathrm{CAlg}^{(2)}_{\mathcal{C}}(\mathcal{C}[\mathfrak{X}], \mathcal{D}) \xrightarrow{\simeq} \mathcal{D}.$$

In particular, the evaluation at $\mathfrak{X}$ gives a categorical equivalence

$$\Sigma\mathrm{Seq}_{\mathrm{B}}(\mathcal{C}) \xrightarrow{\simeq} \mathcal{C}[\mathfrak{X}] \simeq \mathrm{Fun}(\mathsf{FB}, \mathcal{C}).$$

Proof. Remark 1.6 and Proposition A.16 gives an equivalence

$$\mathrm{CAlg}^{(2)}_{\mathcal{C}}(\mathcal{C}[\mathfrak{X}], \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}^{\otimes, L}(\mathcal{S}[\mathfrak{X}], \mathcal{D}),$$

so we are reduced to the case where $\mathcal{C}^{\otimes} = \mathcal{S}^{\times}$. The universal property of the Day convolution symmetric monoidal structure (which follows from the discussion

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[2] Here $\mathrm{CAlg}^{(2)}_{\mathcal{C}}(\mathcal{C}[\mathfrak{X}], \mathcal{D})$ is a shorthand for $\mathrm{CAlg}^{(2)}_{\mathcal{C}}\left(\mathcal{C}[\mathfrak{X}]^{\bigstar}, \mathcal{D}^{\otimes}\right)$

in [Lur17, Corollary 4.8.1.12] and an argument similar to Remark 1.6) gives an equivalence

$$\mathrm{Fun}^{\otimes,L}\left(\mathcal{S}[\mathfrak{X}],\mathcal{D}\right)\xrightarrow{\simeq}\mathrm{Fun}^{\otimes}\left(\mathsf{FB},\mathcal{D}\right).$$

Since $\mathsf{FB}^{\mathrm{op},\amalg}\cong\mathsf{FB}^{\amalg}$ can be identified with the symmetric monoidal envelope (Construction 5.10) of the trivial ∞-operad $\mathsf{Triv}^{\otimes}$ [Lur17, Example 2.1.1.20], we further have an equivalence

$$\mathrm{Fun}^{\otimes}\left(\mathsf{FB},\mathcal{D}\right)\xrightarrow{\simeq}\mathrm{Alg}_{\mathsf{Triv}}\left(\mathcal{D}\right).$$

By [Lur17, Example 2.1.3.5], the evaluation at the unique object of $\mathsf{Triv}^{\otimes}$ gives an equivalence

$$\mathrm{Alg}_{\mathsf{Triv}}\left(\mathcal{D}\right)\xrightarrow{\simeq}\mathcal{D}.$$

The resulting equivalence $\mathrm{Fun}^{\otimes,L}\left(\mathcal{S}[\mathfrak{X}],\mathcal{D}\right)\xrightarrow{\simeq}\mathcal{D}$ is given by the evaluation at $\mathfrak{X}$, and the claim follows. ☐

1.2. **Haugseng's model.** We now turn to Haugseng's model of composition product. In contrast to Brantner's model, Haugseng's model is characterized by a universal property of maps into it. To state it, we must introduce a bit of notation.

Definition 1.13. [Hau22, Definition 4.1.1] We define a category $\mathbf{\Delta}_{\mathbb{F}}$ of **forests** as follows:

(1) Its objects are the finite sequences $S_0\to\cdots\to S_n$ of maps of finite sets in Fin, where $n\geq 0$. We think of such a sequence as a forest whose edges are the elements of $\coprod_{i\geq 0}S_i$ and whose vertices are the elements of $\coprod_{i>1}S_i$.

(2) A morphism $(S_0\to\cdots\to S_n)\to(T_0\to\cdots\to T_m)$ is given by a morphism $\phi\colon[n]\to[m]$ of $\mathbf{\Delta}$ and injective set maps $\{u_i\colon S_i\to T_{\phi(i)}\}_{0\leq i\leq n}$ such that, for each $0\leq i\leq j\leq n$, the square

$$\begin{tikzcd} S_i \arrow[r] \arrow[d] & S_j \arrow[d] \\ T_{\phi(i)} \arrow[r] & T_{\phi(j)} \end{tikzcd}$$

is commutative and cartesian.

In the theory of operads, the category $\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}}$ roughly plays the role that $\mathbf{\Delta}^{\mathrm{op}}$ plays for categories. Informally, the map ϕ decomposes $T_\bullet$ into a bunch of subtrees (with boundaries in $T_{\phi(i)}$ and $T_{\phi(i+1)}$), and the maps u_i carry each of these subtrees into subcorollas of $S_\bullet$ with matching boundaries or discard it entirely.

Definition 1.14. We define the "vertex functor" $V\colon\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}}\to\mathsf{Fin}_*$ by the formula

$$V\left(S_0\to\cdots\to S_n\right)=\left(\coprod_{i>0}S_i\right)_*,$$

where $(-)_*:\mathsf{Fin}\to\mathsf{Fin}_*$ adds a disjoint basepoint. Given a morphism $(\phi,\{u_i\}_i)$ as in Definition 1.13, the map $\left(\coprod_{j>0}T_j\right)_*\to\left(\coprod_{i>0}S_i\right)_*$ is defined as follows: Let $0<i\leq n$ and $s\in S_j$. Then the preimage of s is $\left(\coprod_{\phi(j-1)<k\leq\phi(j)}T_k\right)_{u_i(s)}$, where the subscript indicates the preimage over $u_i\left(s\right)\in T_{\phi(i)}$.

Definition 1.15. [Hau22, Notation 4.2.7] Let $\mathcal{O}^{\otimes}\to\mathbf{\Delta}^{\mathrm{op}}$ be a non-symmetric ∞-operad. A morphism of $\mathcal{O}^{\otimes}\times_{\mathbf{\Delta}^{\mathrm{op}}}\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}}$ is called **operadic inert** if its image in $\mathcal{O}^{\otimes}$ is inert. Given an ∞-operad $\mathcal{C}^{\otimes}\to\mathsf{Fin}_*$, we write $\mathrm{Alg}^{\mathrm{opd}}_{\mathcal{O}^{\otimes}\times_{\mathbf{\Delta}^{\mathrm{op}}}\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}}}\left(\mathcal{C}\right)$ for the full subcategory of $\mathrm{Fun}_{\mathsf{Fin}_*}\left(\mathcal{O}^{\otimes}\times_{\mathbf{\Delta}^{\mathrm{op}}}\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}},\mathcal{C}^{\otimes}\right)$ spanned by the maps carrying operadic inert maps to inert maps of $\mathcal{C}^{\otimes}$.

With these definitions, we can state the universal property of Haugseng's composition product monoidal structure.

Theorem 1.16. [Hau22, Corollary 4.2.9] *Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category compatible with colimits indexed by small ∞-groupoids. There is a monoidal ∞-category $\Sigma\mathrm{Seq}_{\mathrm{H}}\left(\mathcal{C}\right)^{\circ}$ characterized by the equivalence*

$$\mathrm{Alg}^{\mathrm{opd}}_{\mathcal{O}^{\otimes}\times_{\mathbf{\Delta}^{\mathrm{op}}}\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}}}\left(\mathcal{C}\right)\simeq\mathrm{Alg}_{\mathcal{O}}\left(\Sigma\mathrm{Seq}_{\mathrm{H}}\left(\mathcal{C}\right)\right)$$

natural in the non-symmetric ∞-operad $\mathcal{O}^{\otimes}$. Moreover:

1. *The construction is functorial in the variable $\mathcal{C}^{\otimes}$ and symmetric monoidal functors preserving colimits indexed by ∞-groupoids.*
2. *There is an equivalence $\Sigma\mathrm{Seq}_{\mathrm{H}}\left(\mathcal{C}\right)\simeq\mathrm{Fun}\left(\mathsf{FB},\mathcal{C}\right)$ which is functorial in the sense of (1).*

1.3. Main Result.

We can now state the main result of this paper.

Theorem 1.17. *There is an equivalence of monoidal ∞-categories*

$$\Sigma\mathrm{Seq}_{\mathrm{B}}\left(\mathcal{C}\right)^{\circ}\simeq\Sigma\mathrm{Seq}_{\mathrm{H}}\left(\mathcal{C}\right)^{\circ}$$

natural in $\mathcal{C}^{\otimes}\in\mathcal{P}\mathsf{r}\mathcal{SM}$.

In the next two sections (Sections 2 and 3), we establish basic results on monoidal localizations and composition product and Day's convolution product in the model-categorical setting. We then take a closer look at Brantner and Haugseng's models in the ensuing sections (Sections 4 and 5). The proof of Theorem 1.17 will be given in Section 6.

2. Review on Localization

In this section, we recall a few key results and constructions on localization of symmetric or non-symmetric monoidal categories. They will be used in an essential way in the rest of the paper.

First we recall monoidal relative ∞-categories and their localization.

Definition 2.1. [Araa] A **monoidal relative ∞-category** is a pair $(\mathcal{M}^{\otimes},\mathcal{W})$, where $\mathcal{M}^{\otimes}$ is a monoidal ∞-category and $\mathcal{W}\subset\mathcal{M}$ is a subcategory containing all equivalences and are stable under tensor products. Morphisms in $\mathcal{W}$ are called **weak equivalences**. When $\mathcal{W}$ is clear from the context, we often drop it from the notation and say that $\mathcal{M}^{\otimes}$ is a monoidal relative ∞-category.

We let $\mathcal{M}\mathsf{on}\mathcal{R}\mathsf{el}\mathcal{C}\mathsf{at}_{\infty}^{(2)}$ denote the ∞-bicategory of monoidal relative ∞-category; formally, it is the scaled nerve of the sSet^{+}-enriched category whose:

- Objects are monoidal relative ∞-categories.
- Mapping object between a pair of objects $\mathcal{C}^{\otimes},\mathcal{D}^{\otimes}$ is given by the full subcategory $\mathrm{Fun}^{\otimes,\mathrm{rel}}\left(\mathcal{C},\mathcal{D}\right)\subset\mathrm{Fun}^{\otimes}\left(\mathcal{C},\mathcal{D}\right)$ spanned by the monoidal functors that preserve weak equivalences (with equivalences marked).

We think of $\mathcal{M}\mathsf{on}\mathcal{C}\mathsf{at}_{\infty}^{(2)}$ as a full sub ∞-bicategory of $\mathcal{M}\mathsf{on}\mathcal{R}\mathsf{el}\mathcal{C}\mathsf{at}_{\infty}^{(2)}$ via the inclusion $\mathcal{C}^{\otimes}\mapsto(\mathcal{C}^{\otimes},\mathcal{C}^{\simeq})$. The underlying ∞-category of $\mathcal{M}\mathsf{on}\mathcal{R}\mathsf{el}\mathcal{C}\mathsf{at}_{\infty}^{(2)}$ is denoted by $\mathcal{M}\mathsf{on}\mathcal{R}\mathsf{el}\mathcal{C}\mathsf{at}_{\infty}$.

We define symmetric monoidal relative ∞-categories and the associated ∞-bicategory $\mathcal{SMR}\mathsf{el}\mathcal{C}\mathsf{at}_{\infty}^{(2)}$ similarly.

Construction 2.2. Let $(\mathcal{M}^{\otimes},\mathcal{W})$ be a monoidal relative ∞-category. The **monoidal localization** of $(\mathcal{M}^{\otimes},\mathcal{W})$ is a symmetric monoidal ∞-category $\mathcal{N}^{\otimes}$ equipped with a symmetric monoidal functor $\eta_{(\mathcal{M}^{\otimes},\mathcal{W})}\colon\mathcal{M}^{\otimes}\to\mathcal{N}^{\otimes}$ satisfying the following equivalent conditions:

(1) The underlying functor $\mathcal{M} \to \mathcal{N}$ is a localization at $\mathcal{W}$.
(2) For every symmetric monoidal ∞-category $\mathcal{P}$, the functor

$$\eta^*\colon \operatorname{Fun}^{\otimes}(\mathcal{N}, \mathcal{P}) \to \operatorname{Fun}^{\otimes}(\mathcal{M}, \mathcal{P})$$

is fully faithful, and its essential image consists of the functors $\mathcal{M} \to \mathcal{P}$ carrying weak equivalences to equivalences.

(The equivalence of these conditions are proved in [Lur17, Proposition 4.1.7.4].) In this situation, we write $\mathcal{N}^{\otimes} = \mathcal{M}[\mathcal{W}^{-1}]^{\otimes}$.

By [BB24, Corollary A.2.18], the assignment $(\mathcal{M}^{\otimes}, \mathcal{W}) \mapsto \mathcal{M}[\mathcal{W}^{-1}]^{\otimes}$ can be turned into a functor

$$L\colon \mathcal{M}\mathsf{on}\mathcal{R}\mathsf{el}\mathcal{C}\mathsf{at}_{\infty}^{(2)} \to \mathcal{M}\mathsf{on}\mathcal{C}\mathsf{at}_{\infty}^{(2)},$$

and the maps $\eta_{(\mathcal{M}^{\otimes},\mathcal{W})}$ assemble to a natural transformation $\eta\colon \mathrm{id} \Rightarrow \iota \circ L$. For an explicit construction, see Lemma 4.10 below. There is a parallel construction in the symmetric monoidal setting too, which we leave to the reader.

Remark 2.3. By definition, localization of monoidal relative ∞-categories are unique up to equivalence. A stronger uniqueness is also true: Let $\mathcal{C}$ be an ∞-category, and let $F\colon \mathcal{C} \to \mathcal{M}\mathsf{on}\mathcal{R}\mathsf{el}\mathcal{C}\mathsf{at}_{\infty}$ be a functor denoted by $C \mapsto \left(\mathcal{M}_C^{\otimes}, \mathcal{W}_C\right)$. Suppose we are given another functor $\mathcal{N}_{\bullet}^{\otimes}\colon \mathcal{C} \to \mathcal{M}\mathsf{on}\mathcal{C}\mathsf{at}_{\infty}$ and a natural transformation $\alpha\colon \mathcal{M}_{\bullet}^{\otimes} \to \mathcal{N}_{\bullet}^{\otimes}$ of functors $\mathcal{C} \to \mathcal{M}\mathsf{on}\mathcal{C}\mathsf{at}_{\infty}$. If for each $C \in \mathcal{C}$, the map α_C exhibits $\mathcal{N}_C^{\otimes}$ as a monoidal localization of $\mathcal{M}_C^{\otimes}$ at $\mathcal{W}_C$, then there is a natural equivalence

$$L\left(\mathcal{M}_{\bullet}^{\otimes}, \mathcal{W}_{\bullet}\right) \simeq \mathcal{N}_{\bullet}^{\otimes}$$

of functors $\mathcal{C} \to \mathcal{M}\mathsf{on}\mathcal{C}\mathsf{at}_{\infty}$. This follows from the fact that the inclusion

$$\operatorname{Fun}\left(\mathcal{C}, \mathcal{M}\mathsf{on}\mathcal{C}\mathsf{at}_{\infty}\right) \hookrightarrow \operatorname{Fun}\left(\mathcal{C}, \mathcal{M}\mathsf{on}\mathcal{R}\mathsf{el}\mathcal{C}\mathsf{at}_{\infty}\right)$$

has a left adjoint given by postcomposition by L and with unit induced by η. (Alternatively, this follows from [Ram23, Proposition A.11].) In other words, the assignment $C \mapsto \mathcal{M}_C^{\otimes}[\mathcal{W}_C^{-1}]$ can be exteneded to a functor in an essentially unique way. Because of this, we typically denote the functor $L\left(\mathcal{M}_{\bullet}^{\otimes}, \mathcal{W}_{\bullet}\right)$ by $\mathcal{M}_{\bullet}^{\otimes}[\mathcal{W}_{\bullet}^{-1}]$.

We are particularly interested in the localization of symmetric monoidal model categories.

Definition 2.4. A **symmetric monoidal model category** is a model category $\mathbf{M}$ equipped with a closed symmetric monoidal structure, subject to the following pair of conditions:

- (**Cofibrant unit**) The monoidal unit is cofibrant.
- (**Pushout-product**) For every pair of cofibrations $i\colon A \to B$ and $j\colon X \to Y$ in $\mathbf{M}$, their pushout-product

$$i \widehat{\otimes} j\colon (A \otimes Y) \amalg_{A \otimes X} (B \otimes X) \to B \otimes Y$$

is a cofibration. If further i is a trivial cofibration, so is $i \widehat{\otimes} j$.

We write $\mathsf{TractSMMC}$ for the category of tractable symmetric monoidal model categories and left Quillen symmetric monoidal functors. (Recall that a model category is **tractable** if it is locally presentable as a category and has a generating sets of cofibrations and trivial cofibrations with cofibrant domains.)

Let $\mathbf{M}$ be a tractable symmetric monoidal model category. The definition of symmetric monoidal model categories ensures that the full subcategory $\mathbf{M}_{\mathrm{cof}} \subset \mathbf{M}$ of cofibrant objects inherits a symmetric monoidal structure from $\mathbf{M}$ and is a relative symmetric monoidal category. The **underlying symmetric monoidal ∞-category** of $\mathbf{M}$, denoted by $\mathbf{M}_{\infty}^{\otimes}$, is the symmetric monoidal localization of the symmetric monoidal relative category $\mathbf{M}_{\mathrm{cof}}$ at weak equivalences.

Construction 2.2 gives us a functor

$$\mathsf{TractSMMC}^{(2)} \to \mathcal{P}\mathrm{r}\mathcal{S}\mathcal{M}^{(2)},\ \mathbf{M} \mapsto \mathbf{M}^{\otimes}_{\infty}.$$

The main theorem of [Arab] asserts:

Theorem 2.5. *The functor*

$$(-)^{\otimes}_{\infty} : \mathsf{TractSMMC} \to \mathcal{P}\mathrm{r}\mathcal{S}\mathcal{M}$$

is a localization.

3. Composition product and Day convolution from model-categorical viewpoint

In this section, we study composition product and Day convolution from a model-categorical perspective. In Subsection 3.1, we will show that Day's convolution product models ∞-categorical convolution product in the model-categorical setting. In Subsection 3.2, we show that composition product of cofibrant symmetric sequences in a model category behaves well homotopically.

Remark 3.1. For the remainder of this paper, we will adopt the following convention on functor categories of model categories: Let $\mathbf{M}$ be a cofibrantly generated model category, and let $\mathcal{I}$ be a small category. We will always equip $\mathrm{Fun}(\mathcal{I}, \mathbf{M})$ with the *projective* model structure, whose fibrations and weak equivalences are defined pointwise [Hir03, Theorem 11.6.1].

Recall that if I and J are generating sets of cofibrations of $\mathbf{M}$, then the maps $\{\mathcal{I}(i,-) \otimes f \mid f \in I\}$ and $\{\mathcal{I}(i,-) \otimes g \mid g \in J\}$ generate the cofibrations and trivial cofibrations of the projective model structure. (Here $\otimes$ denotes tensor by sets.)

3.1. **Day convolution.** Let $\mathbf{M}$ be a combinatorial symmetric monoidal model category. Given a small symmetric monoidal category $\mathcal{A}$, the category $\mathrm{Fun}(\mathcal{A}, \mathbf{M})$ carries a symmetric monoidal structure, given by the ordinary (1-categorical) Day convolution product $\bigstar_1$. The goal of this subsection is to prove the following proposition, which says that the Day convolution makes $\mathrm{Fun}(\mathcal{A}, \mathbf{M})$ into a symmetric monoidal model category whose underlying symmetric monoidal ∞-category is what we would expect:

Proposition 3.2. *Let* $\mathbf{M}$ *be a combinatorial symmetric monoidal model category with a cofibrant unit, and let* $\mathcal{A}$ *be a small symmetric monoidal category. Equip* $\mathrm{Fun}(\mathcal{A}, \mathbf{M})$ *with the projective model structure and Day's convolution product.*

1. *The category* $\mathrm{Fun}(\mathcal{A}, \mathbf{M})$ *is a symmetric monoidal model category with cofibrant unit.*
2. *There is a coproduct cone*

$$\mathbf{M}^{\otimes}_{\infty} \to \mathrm{Fun}(\mathcal{A}, \mathbf{M})^{\bigstar_1}_{\infty} \xleftarrow{g} \mathrm{Fun}(\mathcal{A}, \mathcal{S})^{\bigstar}$$

in $\mathcal{P}\mathrm{r}\mathcal{S}\mathcal{M}$, *where the left arrow is induced by the symmetric monoidal functor*

$$\mathbf{M} \to \mathrm{Fun}(\mathcal{A}, \mathbf{M}),\ M \mapsto \mathcal{A}(\mathbf{1}, -) \cdot M.$$

3. *For every left Quillen symmetric monoidal functor* $\mathbf{M} \to \mathbf{N}$ *of combinatorial symmetric monoidal model categories, the square*

$$\begin{tikzcd}
\mathbf{M}^{\otimes}_{\infty} \arrow[r] \arrow[d] & \mathbf{N}^{\otimes}_{\infty} \arrow[d] \\
\mathrm{Fun}(\mathcal{A}, \mathbf{M})^{\bigstar_1}_{\infty} \arrow[r] & \mathrm{Fun}(\mathcal{A}, \mathbf{N})^{\bigstar_1}_{\infty}
\end{tikzcd}$$

is cocartesian in $\mathcal{P}\mathrm{r}\mathcal{S}\mathcal{M}$.

(4) We have an equivalence of symmetric monoidal ∞-categories

$$\mathrm{Fun}\left(\mathcal{A},\mathbf{M}\right)_{\infty}^{\bigstar_1}\simeq\mathrm{Fun}\left(\mathcal{A},\mathbf{M}_{\infty}\right)^{\bigstar}.$$

Before turning to the proof of Proposition 3.2, let us prove one of its consequences.

Corollary 3.3. *Let $\mathcal{C}$ be a symmetric monoidal category such that $\mathcal{C}^{\otimes}$ is presentably symmetric monoidal, and let $\mathcal{A}$ be a small symmetric monoidal category. There is a coproduct cone*

$$\mathcal{C}^{\otimes}\xrightarrow{(\{\mathbf{1}\}\hookrightarrow\mathcal{A})_!}\mathrm{Fun}\left(\mathcal{A},\mathcal{C}\right)^{\bigstar_1}\leftarrow\mathrm{Fun}\left(\mathcal{A},\mathcal{S}\right)^{\bigstar}$$

in $\mathcal{P}\mathrm{r}\mathcal{SM}$. In particular, we have an equivalence of symmetric monoidal ∞-categories

$$\mathrm{Fun}\left(\mathcal{A},\mathcal{C}\right)^{\bigstar_1}\simeq\mathrm{Fun}\left(\mathcal{A},\mathcal{C}\right)^{\bigstar}.$$

Proof. We will apply Proposition 3.2 to the trivial model structure on $\mathcal{C}$, whose weak equivalences are the isomorphisms and and whose morphisms are all fibrations. The only nontrivial part is that this model structure is combinatorial.

To see the trivial model structure is combinatorial, take a small regular cardinal κ such that $\mathcal{C}$ is κ-presentable, and let S be a set of representatives of isomorphism classes of κ-compact objects in $\mathcal{C}$. We then set $I=\{A\to B\}_{A,B\in S}$ and $J=\{\mathrm{id}_A\}_{A\in S}$. We claim that I and J generate the classes of cofibrations and trivial cofibrations of $\mathcal{C}$.

It is obvious that every morphism of $\mathcal{C}$ has the right lifting property for the maps in J, and that every isomorphism has the right lifting property for the maps in I. It will therefore suffice to show that if a map $f\colon X\to Y$ has the right lifting property for the maps in I, then f is an isomorphism. For each $A\in S$, the right lifting property for the maps $\emptyset\to A$ and $A\amalg A\to A$ implies that the map $f_*\colon\mathcal{C}\left(A,X\right)\to\mathcal{C}\left(A,Y\right)$ is bijective. Since every object of $\mathcal{C}$ is a colimit of objects in S, it follows that this map is bijective for any object $A\in\mathcal{C}$. Thus f is an isomorphism, as required. □

We now turn to the proof Proposition 3.2, which needs a few preliminaries.

For the next lemma, recall from [Lur09b, Corollary 5.5.3.4 and Theorem 5.5.3.18] that the ∞-category $\mathcal{P}\mathrm{r}^L$ of presentable ∞-categories and left adjoints has small colimits.

Lemma 3.4. *The ∞-category $\mathcal{P}\mathrm{r}^L$ is generated under small colimits by the presheaves on small ∞-categories.*

Proof. Recall that every presentable ∞-category has the form $\mathcal{P}\left(\mathcal{A}\right)\left[S^{-1}\right]$, where $\mathcal{A}$ is a small ∞-category and S is a small set of morphisms of $\mathcal{P}\left(\mathcal{A}\right)$. This localization can be written as a pushout (in $\mathcal{P}\mathrm{r}^L$) of the span

$$\mathcal{P}\left(\coprod_{S}[0]\right)\leftarrow\mathcal{P}\left(\coprod_{f\in S}[1]\right)\xrightarrow{f}\mathcal{P}\left(\mathcal{A}\right),$$

where f is the unique colimit preserving functor determined by the tautological functor $\coprod_{f\in S}[1]\to\mathcal{P}\left(\mathcal{A}\right)$. The claim follows. □

Lemma 3.5. *Let $\mathcal{C}$ be a presentable ∞-category, and let $\mathcal{A}$ be a small ∞-category. Consider the functor* $\mathrm{Fun}\left(\mathcal{A}^{\mathrm{op}},\mathcal{S}\right)\times\mathcal{C}\to\mathrm{Fun}\left(\mathcal{A}^{\mathrm{op}},\mathcal{C}\right)$ *adjoint to the composite*

$$\mathcal{A}^{\mathrm{op}}\times\mathrm{Fun}\left(\mathcal{A}^{\mathrm{op}},\mathcal{S}\right)\times\mathcal{C}\to\mathcal{S}\times\mathcal{C}\to\mathcal{S}\otimes\mathcal{C}\simeq\mathcal{C}$$

The functor Φ induces an equivalence $\alpha_{\mathcal{C}}\colon\mathcal{C}\otimes\mathrm{Fun}\left(\mathcal{A}^{\mathrm{op}},\mathcal{S}\right)\xrightarrow{\simeq}\mathrm{Fun}\left(\mathcal{A}^{\mathrm{op}},\mathcal{C}\right)$ in $\mathcal{P}\mathrm{r}^L$.

Proof. We first observe that the functor $\mathrm{Fun}\left(\mathcal{A}^{\mathrm{op}},-\right):\mathcal{P}\mathsf{r}^L\to\mathcal{P}\mathsf{r}^L$ preserves small colimits. Indeed, using the equivalence $\mathcal{P}\mathsf{r}^L\simeq\left(\mathcal{P}\mathsf{r}^R\right)^{\mathrm{op}}$ of [Lur09b, Corollary 5.5.3.4], we only have to show that the functor $\mathrm{Fun}\left(\mathcal{A}^{\mathrm{op}},-\right):\mathcal{P}\mathsf{r}^R\to\mathcal{P}\mathsf{r}^R$ preserves small limits. This is immediate from [Lur09b, Theorem 5.5.3.18].

Now since $\mathcal{P}\mathsf{r}^L$ is closed symmetric monoidal [Lur17, Remark 4.8.1.18], the functor $\mathcal{P}\left(\mathcal{A}\right)\otimes-:\mathcal{P}\mathsf{r}^L\to\mathcal{P}\mathsf{r}^L$ also preserves small colimits. Therefore, by Lemma 3.4, it will suffice to show that $\alpha_{\mathcal{C}}$ is an equivalence when $\mathcal{C}=\mathcal{P}\left(\mathcal{B}\right)$ for some small ∞-category $\mathcal{B}$.

Let $\mathcal{D}$ be a presentable ∞-category. Precomposing the Yoneda embeddings, we get equivalences

$$\mathrm{Fun}^L\left(\mathcal{P}\left(\mathcal{A}\right)\otimes\mathcal{P}\left(\mathcal{B}\right),\mathcal{D}\right)\xrightarrow{\simeq}\mathrm{Fun}\left(\mathcal{A}\times\mathcal{B},\mathcal{D}\right)\xleftarrow{\simeq}\mathrm{Fun}\left(\mathcal{P}\left(\mathcal{A}\times\mathcal{B}\right),\mathcal{D}\right).$$

This gives us an equivalence $\mathcal{P}\left(\mathcal{A}\right)\otimes\mathcal{P}\left(\mathcal{B}\right)\xrightarrow{\simeq}\mathcal{P}\left(\mathcal{A}\times\mathcal{B}\right)$. Unwinding the definitions, this equivalence is exactly the functor $\alpha_{\mathcal{P}(\mathcal{B})}$, and we are done. ☐

Lemma 3.6. *Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category, and let $X\xrightarrow{f}Z\xleftarrow{g}Y$ be morphisms of commutative algebra objects of $\mathcal{C}^{\otimes}$. The following conditions are equivalent:*

1. *The maps f and g exhibits Z as a coproduct of X and Y in $\mathrm{CAlg}\left(\mathcal{C}\right)$.*
2. *The composite*

$$U\left(X\right)\otimes U\left(Y\right)\xrightarrow{Uf\otimes Ug}U\left(Z\right)\otimes U\left(Z\right)\xrightarrow{\mu}U\left(Z\right)$$

is an equivalence in $\mathcal{C}$, where $U:\mathrm{CAlg}\left(\mathcal{C}\right)\to\mathcal{C}$ denotes the forgetful functor and μ denotes the multiplication of Z.

Proof. Recall that the symmetric monoidal ∞-category $\mathrm{CAlg}\left(\mathcal{C}\right)^{\otimes}$ is cocartesian [Lur17, Proposition 3.2.4.3]. Therefore, by [Lur17, Proposition 2.4.3.16], we can lift $Z\in\mathrm{CAlg}\left(\mathcal{C}\right)$ in an essentially unique way to a commutative algebra object $\overline{Z}$ in $\mathrm{CAlg}\left(\mathcal{C}\right)^{\otimes}$. Using [Lur17, Remark 2.4.3.4], the multiplication map

$$Z\otimes Z\to Z$$

of $\overline{Z}$ can be identified with the codiagonal map $Z\amalg Z\to Z$. So the composite

$$X\otimes Y\xrightarrow{f\otimes g}Z\otimes Z\xrightarrow{\mu}Z$$

is just the map induced by the universal property of coproducts and the maps f and g. It follows that condition (1) is equivalent to the condition that $\mu\circ\left(f\otimes g\right)$ be an equivalence. Since U is conservative and symmetric monoidal [Lur17, Proposition 3.2.4.3], this is equivalent to condition (2). The proof is now complete. ☐

We now arrive at the proof of Proposition 3.2.

Proof of Proposition 3.2. Part (1) is well-known (see, e.g., [BB17, Theorem 4.1]), but it is difficult to find a reference stating this in its exact form, so we record a proof anyway. Recall from Remark 3.1 that if I and J are generating sets of cofibrations and trivial cofibrations of $\mathbf{M}$, then the sets $\{\mathcal{A}\left(a,-\right)\cdot i\}_{a\in\mathcal{A},\,i\in I}$ and $\{\mathcal{A}\left(a,-\right)\cdot j\}_{a\in\mathcal{A},\,j\in J}$ generate the cofibrations and trivial cofibrations of $\mathrm{Fun}\left(\mathcal{A},\mathbf{M}\right)$. Using the isomorphism in $\mathrm{Fun}\left(\mathcal{A},\mathbf{M}\right)$

$$\left(\mathcal{A}\left(a,-\right)\cdot M\right)\star\left(\mathcal{A}\left(b,-\right)\cdot N\right)\cong\mathcal{A}\left(a\otimes b,-\right)\cdot\left(M\otimes N\right),$$

natural in $M,N\in\mathbf{M}$, we deduce that the projective model structure satisfies the pushout-product axiom. Also, the unit object $\mathcal{A}\left(\mathbf{1}_{\mathcal{A}},-\right)\cdot\mathbf{1}_{\mathbf{M}}$ for the Day convolution product is cofibrant, because $\mathbf{1}_{\mathbf{M}}$ is cofibrant. This proves (1).

For part (2), suppose first that there is an equivalence of symmetric monoidal ∞-categories $\mathcal{S}^{\times} \simeq \mathbf{M}^{\otimes}_{\infty}$. The Yoneda embedding

$$\begin{gathered} \mathcal{A}^{\mathrm{op}} \to \mathrm{Fun}\left(\mathcal{A}, \mathbf{M}\right), \\ a \mapsto \mathcal{A}\left(a, -\right) \cdot \mathbf{1} \end{gathered}$$

is symmetric monoidal and takes values in the full subcategory of cofibrant objects. Thus, composing it with the symmetric monoidal localization $\mathrm{Fun}\left(\mathcal{A}, \mathbf{M}\right)^{\bigstar_1}_{\mathrm{cof}} \to \mathrm{Fun}\left(\mathcal{A}, \mathbf{M}\right)^{\bigstar_1}_{\infty}$, we obtain a symmetric monoidal functor $\mathcal{A}^{\mathrm{op}} \to \mathrm{Fun}\left(\mathcal{A}, \mathbf{M}\right)^{\bigstar_1}_{\infty}$. The universal property of the Day convolution product on $\mathrm{Fun}\left(\mathcal{A}, \mathcal{S}\right)$ [Lur17, § 4.8] now gives a symmetric monoidal functor F indicated by the dashed arrow:

$$\begin{tikzcd}
& \mathcal{A}^{\mathrm{op}} \arrow[dl] \arrow[dr] & \\
\mathrm{Fun}(\mathcal{A}, \mathcal{S})^{\bigstar} \arrow[rr, dashed, "F"] & & \mathrm{Fun}(\mathcal{A}, \mathbf{M})^{\bigstar_1}_{\infty}.
\end{tikzcd}$$

We wish to show that F is an equivalence. Since it is symmetric monoidal, it suffices to show that F induces an equivalence between the underlying ∞-categories. Using [Cis19, Theorem 7.9.8], we can identify $\mathrm{Fun}\left(\mathcal{A}, \mathbf{M}\right)_{\infty}$ with $\mathrm{Fun}\left(\mathcal{A}, \mathbf{M}_{\infty}\right)$. Under this identification, the dashed arrow is given by postcomposing the equivalence $\mathcal{S} \simeq \mathbf{M}_{\infty}$. In particular, it is an equivalence, as desired.

For the general case, use [Arab, Corollary ???] to find a symmetric monoidal left Quillen functor $\mathsf{Set}^{\square_{\Sigma}^{\mathrm{op}}} \to \mathbf{M}$, where $\mathsf{Set}^{\square_{\Sigma}^{\mathrm{op}}}$ denotes the tractable symmetric monoidal model category of symmetric cubical sets. This gives us symmetric monoidal left Quillen functors

$$\mathbf{M} \xrightarrow{\phi} \mathrm{Fun}\left(\mathcal{A}, \mathbf{M}\right) \xleftarrow{\psi} \mathrm{Fun}\left(\mathcal{A}, \mathsf{Set}^{\square_{\Sigma}^{\mathrm{op}}}\right),$$

where ϕ is given by $\phi\left(M\right) = \mathcal{A}\left(\mathbf{1}, -\right) \cdot M$. Localizing at weak equivalences, we obtain symmetric monoidal functors

$$\mathbf{M}^{\otimes}_{\infty} \xrightarrow{\phi'} \mathrm{Fun}\left(\mathcal{A}, \mathbf{M}\right)^{\bigstar_1}_{\infty} \xleftarrow{\psi'} \mathrm{Fun}\left(\mathcal{A}, \mathsf{Set}^{\square_{\Sigma}^{\mathrm{op}}}\right)^{\bigstar_1}_{\infty}.$$

From the argument in the previous paragraph, we know that $\mathrm{Fun}\left(\mathcal{A}, \mathsf{Set}^{\square_{\Sigma}^{\mathrm{op}}}\right)^{\bigstar_1}_{\infty}$ is equivalent to $\mathrm{Fun}\left(\mathcal{A}, \mathcal{S}\right)^{\bigstar}$. Therefore, it suffices to show that ϕ' and ψ' form a coproduct cone in $\mathcal{P}\mathsf{r}\mathcal{SM}$.

According to Lemma 3.6, we must show that the composite

$$\begin{aligned} \theta\colon \mathbf{M}_{\infty} \times \mathrm{Fun}\left(\mathcal{A}, \mathsf{Set}^{\square_{\Sigma}^{\mathrm{op}}}\right)_{\infty} &\xrightarrow{\phi' \times \psi'} \mathrm{Fun}\left(\mathcal{A}, \mathbf{M}\right)_{\infty} \times \mathrm{Fun}\left(\mathcal{A}, \mathbf{M}\right)_{\infty} \\ &\xrightarrow{\otimes_{\mathrm{1Day}}} \mathrm{Fun}\left(\mathcal{A}, \mathbf{M}\right)_{\infty} \end{aligned}$$

exhibits $\mathrm{Fun}\left(\mathcal{A}, \mathbf{M}\right)_{\infty}$ as a tensor product of $\mathbf{M}_{\infty}$ and $\mathrm{Fun}\left(\mathcal{A}, \mathsf{Set}^{\square_{\Sigma}^{\mathrm{op}}}\right)_{\infty}$ in $\mathcal{P}\mathrm{r}^{L}$. Using [Cis19, Theorem 7.9.8], we can identify $\mathrm{Fun}\left(\mathcal{A}, \mathbf{M}\right)_{\infty}$ with $\mathrm{Fun}\left(\mathcal{A}, \mathbf{M}_{\infty}\right)$, and $\mathrm{Fun}\left(\mathcal{A}, \mathsf{Set}^{\square_{\Sigma}^{\mathrm{op}}}\right)_{\infty}$ with $\mathrm{Fun}\left(\mathcal{A}, \mathcal{S}\right)$. Under these identifications, the map θ is adjoint to the composite

$$\begin{aligned} \mathbf{M}_{\infty} \times \mathrm{Fun}\left(\mathcal{A}, \mathcal{S}\right) \times \mathcal{A} &\xrightarrow{\mathrm{id} \times \mathrm{ev}} \mathbf{M}_{\infty} \times \mathcal{S} \\ &\to \mathbf{M}_{\infty} \otimes \mathcal{S} \\ &\xrightarrow{\simeq} \mathbf{M}_{\infty}. \end{aligned}$$

So the claim follows from Lemma 3.5.

Part (3) follows from the argument in part (2) and the pasting law of pushouts [Lur09b, Lemma 4.4.2.1]. Part (4) is a consequence of (3). The proof is now complete. □

3.2. **Composition Product.** Let $\mathbf{M}$ be a symmetric monoidal model category. The composition product monoidal structure on $\Sigma\mathrm{Seq}(\mathbf{M})$ is generally *not* closed, because composition product may not preserve small colimits in the first variable. In particular, $\Sigma\mathrm{Seq}(\mathbf{M})$ is generally not a monoidal model category. The goal of this subsection is to show that the composition product still behaves homotopically for cofibrant symmetric sequences. More precisely, we prove the following proposition:

Proposition 3.7. *Let $\mathbf{M}$ be a cofibrantly generated symmetric monoidal model category that admits generating sets of cofibrations and trivial cofibrations whose domains are cofibrant. Then:*

1. *For every projectively cofibrant symmetric sequence X, the functor*
$$X\circ - \colon \Sigma\mathrm{Seq}(\mathbf{M}) \to \Sigma\mathrm{Seq}(\mathbf{M})$$
is left Quillen.
2. *For every projectively cofibrant symmetric sequence Y, the functor*
$$-\circ Y \colon \Sigma\mathrm{Seq}(\mathbf{M}) \to \Sigma\mathrm{Seq}(\mathbf{M})$$
preserves weak equivalences of projectively cofibrant objects.

For the proof of Proposition 3.7, we need some preliminaries.

Lemma 3.8. *Let $\mathbf{M}$ be a symmetric monoidal model category, and let $\{f_i\colon X_i \to Y_i\}_{i=1,2}$ be cofibrations of $\mathbf{M}$. If X_1 and X_2 are cofibrant, the map*
$$f_1\otimes f_2\colon X_1\otimes X_2 \to Y_1\otimes Y_2$$
is a cofibration. If further one of f_1 or f_2 is a weak equivalence, so is $f_1\otimes f_2$.

Proof. We will show that $f_1\otimes f_2$ is a cofibration; the latter claim can be proved similarly. We can factor $f_1\otimes f_2$ as
$$X_1\otimes X_2 \xrightarrow{f_1\otimes \mathrm{id}} Y_1\otimes X_2 \xrightarrow{\mathrm{id}\otimes f_2} Y_1\otimes Y_2.$$
The map $f_1\otimes \mathrm{id}$ is a cofibration by the pushout-product axiom (applied to the cofibrations f_1 and $\emptyset\to X_2$). Likewise, $\mathrm{id}\otimes f_2$ is a cofibration. Hence $f_1\otimes f_2$ is the composite of two cofibrations, and we are done. □

To state our next lemma, we need to introduce some notation.

Notation 3.9. For each $A\in\mathsf{FB}$, we let Σ_A denote the automorphism group of A, and write $B\Sigma_A\subset\mathsf{FB}$ for the full subcategory spanned by A.

It will also be useful to have the following alternative notation for $B\Sigma_A$:

Notation 3.10. For each $r\geq 0$, we let $\mathsf{FB}(r)\subset\mathsf{FB}$ denote the full subgroupoid spanned by the (unique) object of cardinality r.

Lemma 3.11. *Let $\mathbf{M}$ be a tractable symmetric monoidal model category, let X be a projectively cofibrant symmetric sequence in $\mathbf{M}$, and let $A\in\mathsf{FB}$ be a finite set. For each $r\geq 0$, the functor*
$$\begin{aligned}F_r\colon \Sigma\mathrm{Seq}(\mathbf{M}) &\to \mathrm{Fun}(B\Sigma_A,\mathbf{M})\\ Y &\mapsto \mathrm{colim}_{f\colon A\to B\in\mathsf{FB}(r)\times_{\mathsf{Fin}}\mathsf{Fin}_{A/}}\left(\bigotimes_{b\in B} X\left(f^{-1}(b)\right)\right)\otimes Y(B)\end{aligned}$$
is left Quillen.

Proof. Since $\mathbf{M}$ is tractable, there are generating sets I and J of cofibrations and trivial cofibrations with cofibrant domains. We define sets I_Σ and J_Σ of morphisms of $\Sigma\mathrm{Seq}\left(\mathbf{M}\right)$ by

$$\begin{aligned} I_\Sigma &= \{S_!\alpha \mid S \in \mathsf{FB}, \alpha \in I\},\\ J_\Sigma &= \{S_!\alpha \mid S \in \mathsf{FB}, \alpha \in J\}, \end{aligned}$$

where $S_! = \mathsf{FB}\left(S,-\right) \otimes - : \mathbf{M} \to \Sigma\mathrm{Seq}\left(\mathbf{M}\right)$ denotes the left adjoint to the evaluation at S. As we saw in Remark 3.1, the sets I_Σ and J_Σ form generating sets of cofibrations and trivial cofibrations of $\Sigma\mathrm{Seq}\left(\mathbf{M}\right)$. Since F_r preserves small colimits, it suffices to show that F_r carries morphisms in I_Σ to cofibrations and morphisms in J_Σ to trivial cofibrations. In what follows, we will focus on the case of I_Σ; the proof for J_Σ can be treated similarly.

Let S be a finite set, and let $\alpha : P \to P'$ be a morphism in I. We wish to show that the morphism $F_r\left(S_!\left(\alpha\right)\right)$ is a cofibration. If the cardinality of S is not equal to r, then $F_r\left(S_!\left(\alpha\right)\right)$ is an isomorphism, and we are done. If S has exactly r elements, then we can identify $F_r\left(S_!\left(\alpha\right)\right)$ with the map

$$\coprod_{f\in\mathsf{Fin}(A,S)}\left(\bigotimes_{s\in S} X\left(f^{-1}\left(s\right)\right)\right)\otimes P \to \coprod_{f\in\mathsf{Fin}(A,S)}\left(\bigotimes_{s\in S} X\left(f^{-1}\left(s\right)\right)\right)\otimes P',$$

Therefore, it suffices to show that the functor

$$\begin{aligned} \Phi\colon \Sigma\mathrm{Seq}\left(\mathbf{M}\right)^S \times \mathbf{M} &\to \mathrm{Fun}\left(B\Sigma_A, \mathbf{M}\right),\\ \left(\left(X_s\right)_{s\in S}, N\right) &\mapsto \coprod_{f\in\mathsf{Fin}(A,S)}\left(\bigotimes_{s\in S} X_s\left(f^{-1}\left(s\right)\right)\right)\otimes N \end{aligned}$$

carries cofibrations to cofibrations. As before, we can check this on the level of generating cofibrations. In other words, it suffices to show that, for each collection $\left(T_s\right)_{s\in S}$ of objects in FB, the composite

$$\Psi\colon \mathbf{M}^S \times \mathbf{M} \xrightarrow{\left(\prod_{s\in S}(T_s)_!\right)\times\mathrm{id}_{\mathbf{M}}} \Sigma\mathrm{Seq}\left(\mathbf{M}\right)^S \times \mathbf{M} \xrightarrow{\Phi} \mathrm{Fun}\left(B\Sigma_A, \mathbf{M}\right)$$

preserves cofibrations. Unwinding the definitions, the functor Ψ is given by

$$\Psi\left(M_1,\dots,M_r,N\right) = \mathsf{FB}\left(\coprod_{s\in S} T_s, A\right) \otimes \left(\left(\bigotimes_{s\in S} M_s\right)\otimes N\right).$$

Since $\mathsf{FB}\left(\coprod_{s\in S} T_s, A\right)$ is a free Σ_A-set, the functor $\mathsf{FB}\left(\coprod_{s\in S} T_s, A\right) \otimes - : \mathbf{M} \to \mathrm{Fun}\left(B\Sigma_A, \mathbf{M}\right)$ is left Quillen. Thus, we are reduced to showing that the functor

$$\bigotimes\colon \mathbf{M}^S \times \mathbf{M} \to \mathbf{M}$$

carries cofibrations of cofibrant objects to cofibrations. But this is the content of Lemma 3.8, and we are done. □

Lemma 3.12. *Let $\mathbf{M}$ be a tractable symmetric monoidal model category, and let X and Y be projectively cofibrant symmetric sequence in $\mathbf{M}$. For each finite set A and each $r \geq 0$, the colimit cone for*

$$\mathrm{colim}_{f:A\to B\in\mathsf{FB}\times_{\mathsf{Fin}}\mathsf{Fin}_{A/}}\left(\bigotimes_{b\in B} X\left(f^{-1}\left(b\right)\right)\right)\otimes Y\left(B\right)$$

is a homotopy colimit cone (i.e., its image in $\mathbf{M}[\mathrm{weq}^{-1}]$ is a colimit cone). Moreover, this colimit is cofibrant.

Proof. The final claim is a consequence of Lemma 3.11, because cofibrant objects are stable under coproducts. For the former, let $G_{X,Y}\colon \mathsf{FB}\times_{\mathsf{Fin}}\mathsf{Fin}_{A/}\to \mathbf{M}$ denote the diagram

$$(f\colon A\to B)\mapsto \left(\bigotimes_{b\in B} X\left(f^{-1}(b)\right)\right)\otimes Y(B),$$

and let $p\colon \mathsf{FB}\times_{\mathsf{Fin}}\mathsf{Fin}_{A/}\to \mathsf{FB}$ denote the projection. The left Kan extension $\operatorname{Lan}_p G_{X,Y}$ is given by the formula

$$\operatorname{Lan}_p G_{X,Y}(B)=\coprod_{f\in\mathsf{Fin}(A,B)}\bigotimes_{b\in B} X\left(f^{-1}(b)\right)\otimes Y(B).$$

Since coproducts of cofibrant objects are homotopy coproducts, [Lur25, Tag 02ZM] shows that this is a homotopy left Kan extension. (Note that X and Y are objectwise cofibrant, being projectively cofibrant). By the transitivity of Kan extensions, we have $\operatorname{colim}_{\mathsf{FB}_{A/}}\cong\operatorname{colim}_{\mathsf{FB}}\circ\operatorname{Lan}_p$. Therefore, it suffices to show that the colimit cone for $\operatorname{colim}_{\mathsf{FB}}\operatorname{Lan}_p G_{X,Y}$ is a homotopy colimit diagram. We prove this by showing that $\operatorname{Lan}_p G_{X,Y}\in\operatorname{Fun}(\mathsf{FB},\mathbf{M})$ is projectively cofibrant. In fact, we will prove more strongly that the functor

$$\begin{aligned}\Sigma\mathrm{Seq}(\mathbf{M})&\to\operatorname{Fun}(\mathsf{FB},\mathbf{M}),\\ Y&\mapsto(\operatorname{Lan}_p G_{X,Y})\end{aligned}$$

is left Quillen.

Let $B\in\mathsf{FB}$ be an arbitrary object. We must show that the functor

$$\begin{aligned}\Phi\colon\Sigma\mathrm{Seq}(\mathbf{M})&\to\operatorname{Fun}(B\Sigma_B,\mathbf{M}),\\ Y&\mapsto\coprod_{f\in\mathsf{Fin}(A,B)}\bigotimes_{b\in B}X\left(f^{-1}(b)\right)\otimes Y(B)\end{aligned}$$

is left Quillen. By Remark 3.1, it suffices to show that for each cofibration (resp. trivial cofibration) $\alpha\colon M\to M'$ in $\mathbf{M}$ and each $S\in\mathsf{FB}$, the map $\Phi(\mathsf{FB}(S,-)\otimes\alpha)$ is a cofibration (resp. trivial cofibration). We will focus on the case of cofibrations, because the case of trivial cofibrations can be dealt with similarly.

If $|S|\neq|B|$, then $\Phi(\mathsf{FB}(S,-)\otimes\alpha)$ is an isomorphism, and we are done. If $|S|=|B|$, let $\Pi(A,B)$ denote the set of connected components of $B\Sigma_B\times_{\mathsf{Fin}}\mathsf{Fin}_{A/}$. For each $\pi\in\Pi(A,B)$, we will write $\mathsf{Fin}^\pi(A,B)\subset\mathsf{Fin}(A,B)$ for the set of maps $A\to B$ lying in π. We also choose a representative f_π of each $\pi\in\Pi(A,B)$ and set $X_\pi=\bigotimes_{b\in B}X\left(f_\pi^{-1}(b)\right)$. We then have a Σ_B-equivariant isomorphism

$$\Phi(\mathsf{FB}(S,-)\otimes M)\cong\coprod_{\pi\in\Pi(A,B)}\left(\left(\mathsf{Fin}^\pi(A,B)\times\mathsf{FB}(S,B)\right)\otimes X_\pi\otimes M\right)$$

natural in M. Since the Σ_B-set $\mathsf{Fin}^\pi(A,B)\times\mathsf{FB}(S,B)$ is free, this isomorphism tells us that $\Phi(\mathsf{FB}(S,-)\otimes\alpha)$ can be identified with the image of a cofibration of $\mathbf{M}$ under the left Kan extension functor $\mathbf{M}\to\operatorname{Fun}(B\Sigma_B,\mathbf{M})$. Since the latter is a left Quillen functor, we have shown that $\Phi(\mathsf{FB}(S,-)\otimes\alpha)$, as desired. ☐

We now arrive at the proof of Proposition 3.7.

Proof of Proposition 3.7. For (1), we must show that, for each finite set A, the composite

$$F\colon\Sigma\mathrm{Seq}(\mathbf{M})\xrightarrow{X\circ-}\Sigma\mathrm{Seq}(\mathbf{M})\xrightarrow{\text{restriction}}\operatorname{Fun}(B\Sigma_A,\mathbf{M})$$

is left Quillen. This follows from Lemma 3.11, because F is the coproduct of the functors $\{F_r\}_{r\geq 0}$. Part (2) follows from 3.12, and we are done. ☐

4. Closer look at Brantner's model

The goal of this section is to characterize Brantner's model by a universal property. As we saw in Proposition 3.7, for every tractable symmetric monoidal model category $\mathbf{M}$, the full subcategory $\Sigma\mathrm{Seq}\left(\mathbf{M}\right)_{\mathrm{cof}} \subset \Sigma\mathrm{Seq}\left(\mathbf{M}\right)$ of projectively cofibrant objects is a monoidal category itself, and its tensor product preserves weak equivalences in each variable. In particular, it admits a monoidal localization. Our main theorem asserts that Brantner's model is naturally equivalent to this monoidal localization:

Theorem 4.1. *There is a natural equivalence*

$$\Sigma\mathrm{Seq}\left(\mathbf{M}\right)_{\mathrm{cof}}[\mathrm{weq}^{-1}]^{\circ} \simeq \Sigma\mathrm{Seq}_{\mathrm{B}}\left(\mathbf{M}_{\infty}\right)^{\circ}$$

of functors $\mathsf{TractSMMC} \to \mathcal{M}\mathrm{on}\widehat{\mathcal{C}\mathrm{at}}_{\infty}$.

The remainder of this section is devoted to the proof of Theorem 4.1.

Notation 4.2. Let $\mathbf{M}$ be a combinatorial symmetric monoidal model category. We write $\mathbf{M}[\mathcal{X}] = \mathrm{Fun}\left(\mathsf{FB}, \mathbf{M}\right)$, and write $\bigstar$ for the (ordinary) Day convolution product in $\mathbf{M}[\mathcal{X}]$. We also write $\mathcal{X} = (\emptyset, \mathbf{1}, \emptyset, \emptyset, \dots) \in \mathbf{M}[\mathcal{X}]$ for the unit symmetric sequence, and $i_{\mathbf{M}}\colon \mathbf{M} \to \mathbf{M}[\mathcal{X}]$ for the functor $M \mapsto (M, \emptyset, \emptyset, \dots)$. (This set of notation is justified by Corollary 3.3.)

Definition 4.3. We let $\mathsf{TractSMMC}_{\emptyset} \subset \mathsf{TractSMMC}$ denote the full subcategory spanned by the objects $\mathbf{M}$ which admits *exactly one* initial object $\emptyset$.

Remark 4.4. Here is the motivation for Definition 4.3. In the rest of this section, we will frequently be considering diagrams of the form

$$\begin{tikzcd}
\mathbf{M} \arrow[r, "F"] \arrow[d, hook, "i_{\mathbf{M}}"'] & \mathbf{N} \arrow[d, hook, "i_{\mathbf{N}}"] \\
\mathbf{M}[\mathcal{X}] \arrow[r, "F_*"'] & \mathbf{N}[\mathcal{X}],
\end{tikzcd}$$

where F is a left Quillen symmetric monoidal functor and $i_{\mathbf{M}}, i_{\mathbf{N}}$ are defined in Notation 1.8. The diagram may not commute on the nose, but it does when $\mathbf{N}$ has only one initial object.

Remark 4.5. Theorem 2.5 remains valid if we replace $\mathsf{TractSMMC}$ by $\mathsf{TractSMMC}_{\emptyset}$. Indeed, the argument of [Arab, ???] shows that the $(2,1)$-categorical enhancements of $\mathsf{TractSMMC}_{\emptyset}$ and $\mathsf{TractSMMC}$ are localizations of $\mathsf{TractSMMC}$ and $\mathsf{TractSMMC}_{\emptyset}$, and these $(2,1)$-categorical enhancements are equivalent. Since the maps inverted by these localizations are contained in the class of symmetric monoidal left Quillen equivalences, this implies that the inclusion $\mathsf{TractSMMC}_{\emptyset} \hookrightarrow \mathsf{TractSMMC}$ induces an equivalence upon localizing at symmetric monoidal left Quillen equivalences.

Construction 4.6. The category $\mathsf{TractSMMC}_{\emptyset}$ admits a natural 2-categorical enhancement $\mathsf{TractSMMC}_{\emptyset}^{(2)}$. Explicitly, given a pair of objects $\mathbf{M}$ and $\mathbf{N}$, the mapping categories from $\mathbf{M}$ to $\mathbf{N}$ are given by the full subcategory $\mathrm{Fun}^{\otimes, LQ}\left(\mathbf{M}, \mathbf{N}\right) \subset \mathrm{Fun}^{\otimes}\left(\mathbf{M}, \mathbf{N}\right)$ of left Quillen symmetric monoidal functors.

If $\mathbf{M}$ and $\mathbf{N}$ are equipped with a left Quillen symmetric monoidal functors $F\colon \mathbf{A} \to \mathbf{M}$ and $G\colon \mathbf{A} \to \mathbf{N}$, we define another category $\mathrm{Fun}_{\mathbf{A}/}^{\otimes, LQ}\left(\mathbf{M}, \mathbf{N}\right)$ by the (strict) pullback

$$\begin{tikzcd}
\mathrm{Fun}_{\mathbf{A}/}^{\otimes, LQ}(\mathbf{M}, \mathbf{N}) \arrow[r] \arrow[d] & \mathrm{Fun}^{\otimes, LQ}(\mathbf{M}, \mathbf{N}) \arrow[d] \\
\{*\} \arrow[r] & \mathrm{Fun}^{\otimes, LQ}(\mathbf{A}, \mathbf{N}).
\end{tikzcd}$$

We will often consider the category $\mathrm{Fun}^{\otimes,LQ}_{\mathbf{A}/}(\mathbf{M},\mathbf{N})$ when F is injective on objects and morphisms; in this case, the above square is homotopy cartesian.

Proposition 4.7. *For every morphism $i\colon \mathbf{M}\to\mathbf{N}$ in $\mathsf{TractSMMC}_{\emptyset}$, the evaluation at the unit symmetric sequence $\mathfrak{X}\in\mathbf{M}[\mathfrak{X}]$ induces a categorical equivalence*

$$\theta\colon\ \mathrm{Fun}^{\otimes,LQ}_{\mathbf{M}/}(\mathbf{M}[\mathfrak{X}],\mathbf{N})\xrightarrow{\simeq}\mathbf{N}_{\mathrm{cof}}.$$

Proof. Using Proposition 1.12 and Corollary 3.3, we deduce that the evaluation at $\mathfrak{X}$ gives an equivalence of categories

$$\theta'\colon\ \mathrm{Fun}^{\otimes,L}_{\mathbf{M}/}(\mathbf{M}[\mathfrak{X}],\mathbf{N})\xrightarrow{\simeq}\mathbf{N}.$$

Therefore, it suffices to show that a functor $F\in\mathrm{Fun}^{\otimes,L}_{\mathbf{M}/}(\mathbf{M}[\mathfrak{X}],\mathbf{N})$ is left Quillen if and only if $\theta'(F)$ is cofibrant.

Necessity is obvious, because E is cofibrant. For sufficiency, suppose that $\theta'(F)$ is cofibrant. We must show that F preserves cofibrations and trivial cofibrations. We will show that it preserves cofibrations; the argument for trivial cofibrations will be similar. Recall from Remark 3.1 that the class of cofibrations of $\mathrm{Fun}(\mathsf{FB},\mathbf{M})$ is generated by the maps of the form

$$f\colon \mathsf{FB}(S,-)\cdot M\to\mathsf{FB}(S,-)\cdot M',$$

where $S\in\mathsf{FB}$ is a finite set and $M\to M'$ is a cofibration of $\mathbf{M}$. We can rewrite f as

$$\mathfrak{X}^{\bigstar S}\star i_{\mathbf{M}}(M)\to\mathfrak{X}^{\bigstar S}\star i_{\mathbf{M}}(M').$$

Since F is symmetric monoidal and is compatible with the restrictions to $\mathbf{M}$, the morphism $F(f)$ can be identified with the map

$$F(\mathfrak{X})^{\otimes S}\otimes i(M)\to F(\mathfrak{X})^{\otimes S}\otimes i(M').$$

This is a cofibration because i is left Quillen and $\theta'(F)=F(\mathfrak{X})$ is cofibrant. Hence F carries generating cofibrations to cofibrations. It follows that F carries all cofibrations to cofibrations, as claimed. ☐

Notation 4.8. We write:

- $\widehat{\mathsf{MonCat}}$ for the category of large monoidal categories and monoidal functors;
- $\widehat{\mathsf{BiCat}}$ for the category of large bicategories;
- $B\colon\widehat{\mathsf{MonCat}}\hookrightarrow\widehat{\mathsf{BiCat}}$ for the inclusion; and
- $\widehat{\mathcal{B}i\mathcal{C}at}_{\infty}$ for the ∞-category of large ∞-bicategories.

We write $\int^{\mathsf{TractSMMC}_{\emptyset}}B\Sigma\mathrm{Seq}(-)^{\circ}_{\mathrm{cof}}$ for the full sub ∞-bicategory of the fiber product

$$\mathrm{Fun}^{\mathrm{sc}}\left([1],\mathsf{TractSMMC}^{(2)}_{\emptyset}\right)\times_{\mathrm{Fun}\left(\{0\},\mathsf{TractSMMC}^{(2)}_{\emptyset}\right)}\mathsf{TractSMMC}_{\emptyset}$$

spanned by the objects $\{i_{\mathbf{M}}\colon\mathbf{M}\to\mathbf{M}[\mathfrak{X}]\}_{\mathbf{M}\in\mathsf{TractSMMC}_{\emptyset}}$.

Notation 4.8 is justified by the following proposition:

Proposition 4.9. *Let $p\colon\int^{\mathsf{TractSMMC}_{\emptyset}}B\Sigma\mathrm{Seq}(-)^{\circ}_{\mathrm{cof}}\to\mathsf{TractSMMC}_{\emptyset}$ be as in Notation 4.8.*

(1) For every map $F\colon\mathbf{M}\to\mathbf{N}$ in $\mathsf{TractSMMC}_{\emptyset}$, the square

$$\begin{tikzcd}
\mathbf{M} \arrow[r, "F"] \arrow[d, hook, "i_{\mathbf{M}}"'] & \mathbf{N} \arrow[d, hook, "i_{\mathbf{N}}"] \\
\mathbf{M}[\mathfrak{X}] \arrow[r, "F_*"'] & \mathbf{N}[\mathfrak{X}]
\end{tikzcd}$$

is p-cocartesian.

(2) *The functor p is a cocartesian fibration of ∞-bicategories which straightens to the composite*

$$\Phi\colon \mathsf{TractSMMC}_{\emptyset} \xrightarrow{\left(\Sigma\mathrm{Seq}(-)_{\mathrm{cof}},\circ\right)} \widehat{\mathsf{MonCat}} \xrightarrow{B} \widehat{\mathsf{BiCat}} \hookrightarrow \mathcal{B}\mathsf{i}\widehat{\mathcal{C}\mathsf{at}}_{\infty}.$$

Proof. We start with (1). Let $\mathcal{X}$ denote the full sub 2-category of

$$\widetilde{\mathcal{X}} = 2\,\mathrm{Fun}\left([1], \mathsf{TractSMMC}_{\emptyset}^{(2)}\right) \times_{2\,\mathrm{Fun}\left(\{0\},\mathsf{TractSMMC}_{\emptyset}^{(2)}\right)} \mathsf{TractSMMC}_{\emptyset}$$

spanned by the objects $\{i_{\mathbf{M}}\colon \mathbf{M} \to \mathbf{M}[\mathfrak{X}]\}_{\mathbf{M}\in\mathsf{TractSMMC}_{\emptyset}}$, where $2\,\mathrm{Fun}\,(-,-)$ denotes the category of 2-functors and 2-natural transformations (i.e., Cat-enriched natural transformations). By part (2) of Proposition A.16, the functor $\mathcal{X} \to \int^{\mathsf{TractSMMC}_{\emptyset}} B\Sigma\mathrm{Seq}\,(-)^{\circ}_{\mathrm{cof}}$ is an equivalence. It will therefore suffice to show that the square in the statement is cocartesian for the projection $\widetilde{\mathcal{X}} \to \mathsf{TractSMMC}_{\emptyset}$.

Given a left Quillen symmetric monoidal functor $G\colon \mathbf{P} \to \mathbf{Q}$, the mapping category of $\widetilde{\mathcal{X}}$ is given by

$$\mathcal{X}\,(i_{0,\mathbf{M}}, G) = \coprod_{\mathbf{M}\to\mathbf{P}\in\mathrm{Fun}^{\otimes,LQ}(\mathbf{M},\mathbf{P})} \mathrm{Fun}^{\otimes,LQ}_{\mathbf{M}/}\left(\mathbf{M}[\mathfrak{X}], \mathbf{Q}\right).$$

Thus, our goal is to show that, for each left Quillen symmetric monoidal functor $\mathbf{N} \to \mathbf{P}$, the map

$$\mathrm{Fun}^{\otimes,LQ}_{\mathbf{N}/}\left(\mathbf{N}[\mathfrak{X}], \mathbf{Q}\right) \to \mathrm{Fun}^{\otimes,LQ}_{\mathbf{M}/}\left(\mathbf{M}[\mathfrak{X}], \mathbf{Q}\right)$$

is an equivalence. This follows from Proposition 4.7, which identifies this map with $\mathrm{id}\colon \mathbf{Q}_{\mathrm{cof}} \to \mathbf{Q}_{\mathrm{cof}}$.

We next turn to (2). Part (1) implies that p is a cocartesian fibration. To identify its straightening, let $\int\Phi \to \mathsf{TractSMMC}_{\emptyset}$ denote the bicategorical Grothendieck construction (Variant A.25) of Φ. Explicitly, objects of $\int\Phi$ are the objects of $\mathsf{TractSMMC}_{\emptyset}$, and the mapping categories are defined by

$$\left(\int\Phi\right)(\mathbf{M},\mathbf{N}) = \coprod_{\mathbf{M}\to\mathbf{N}\in\mathrm{Fun}^{\otimes,LQ}(\mathbf{M},\mathbf{N})} \Sigma\mathrm{Seq}\,(\mathbf{N})_{\mathrm{cof}}.$$

So a 1-morphism $\mathbf{M} \to \mathbf{N}$ is a pair (F, X), where $F\colon \mathbf{M} \to \mathbf{N}$ is a left Quillen symmetric monoidal functor, and X is a projectively cofibrant symmetric sequence in $\mathbf{N}$. Composition of such are defined by

$$(G, Y) \circ (F, X) = (GF, Y \circ (F_{*}X)).$$

2-morphisms are defined and composed in a similar way. By Variant A.25, it suffices to produce a strictly unitary equivalence $\mathcal{X} \xrightarrow{\simeq} \int\Phi$ of bicategories. Such an equivalence is obtained from Proposition 4.7 by arguing as in Proposition 1.1. ☐

Lemma 4.10. *We can construct the symmetric monoidal localization functor $L\colon \mathcal{SMR}\mathsf{el}\mathcal{C}\mathsf{at}_{\infty}^{(2)} \to \mathcal{SMC}\mathsf{at}_{\infty}^{(2)}$ and the natural transformation $\eta = \{(\mathcal{C}^{\otimes}, \mathcal{W}) \to \mathcal{C}[\mathcal{W}^{-1}]^{\otimes}\}_{(\mathcal{C}^{\otimes},\mathcal{W})\in\mathcal{SMR}\mathsf{el}\mathcal{C}\mathsf{at}_{\infty}^{(2)}}$ of Construction 2.2 so that they satisfy the following properties:*

(1) *L is the scaled nerve of an sSet^{+}-enriched functor.*
(2) *η is the scaled nerve of an sSet^{+}-enriched natural transformation.*
(3) *If $(\mathcal{C}^{\otimes}, \mathcal{W}_{\mathcal{C}}) \to (\mathcal{D}^{\otimes}, \mathcal{W}_{\mathcal{D}})$ is a morphism of symmetric monoidal relative categories such that $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a monomorphism of simplicial sets, then the map $\mathcal{C}[\mathcal{W}_{\mathcal{C}}^{-1}]^{\otimes} \to \mathcal{D}[\mathcal{W}_{\mathcal{D}}^{-1}]^{\otimes}$ is a monomorphism of simplicial sets.*

Proof. The key ingredient is the enriched version of Quillen's small object argument [Rie14, Theorem 13.2.1], which we now recall. Let $\mathsf{Fin}_{*}^{\sharp}$ denote the marked simplicial set obtained from (the nerve of) Fin_{*} by marking all of its morphisms.

According to [Lur17, Variant 2.1.4.13, Remark B.2.5], the category $\mathsf{sSet}^+_{/\mathsf{Fin}_*^\sharp}$ admits a combinatorial sSet^+-enriched model structure with the following properties:

- Its cofibrations are the monomorphisms; and
- Its fibrant objects are the symmetric monoidal ∞-categories with cocartesian edges marked.

Now fix a generating set J of trivial cofibrations of $\mathsf{sSet}^+_{/\mathsf{Fin}_*}$, and choose a regular cardinal κ such that the domain and codomain of the maps in J are all κ-compact. We define a κ-sequence $\mathrm{id} = L_0 \Rightarrow L_1 \Rightarrow \cdots$ of enriched endofunctors of $\mathsf{sSet}^+_{/\mathsf{Fin}_*}$ and enriched natural transformations inductively as follows: For each limit ordinal λ, we set $L_\lambda = \operatorname{colim}_{\alpha<\lambda} L_\alpha$. Assuming that L_α has been defined for some $\alpha < \kappa$, we define $L_{\alpha+1}$ by the pushout

$$\begin{tikzcd}
\coprod_{(j\colon A_j \to B_j)\in J}[A_j, L_\alpha(X)] \times A_j \arrow[r] \arrow[d] & L_\alpha(X) \arrow[d] \\
\coprod_{(j\colon A_j \to B_j)\in J}[A_j, L_\alpha(X)] \times B_j \arrow[r] & L_{\alpha+1}(X),
\end{tikzcd}$$

where $[-,-]$ denotes the mapping objects of the sSet^+-enriched category $\mathsf{sSet}^+_{/\mathsf{Fin}_*}$. We then set $L_\kappa = \operatorname{colim}_{\alpha<\kappa} L_\alpha$ and write $\eta_\kappa\colon \mathrm{id}_{\mathsf{sSet}^+_{/\mathsf{Fin}_*}} \Rightarrow L_\kappa$ for the resulting enriched natural transformation. By construction, the components of η_κ are trivial cofibrations of $\mathsf{sSet}^+_{/\mathsf{Fin}_*}$, and L takes values in the full subcategory of fibrant objects. Moreover, L preserves monomorphisms because the functor $[A,-]$ preserves monomorphisms for all $A \in \mathsf{sSet}^+_{/\mathsf{Fin}_*}$.

To relate this construction to the setting at hand, let $\mathcal{X}$ and $\mathcal{Y}$ denote the sSet^+-enriched categories defining $\mathcal{SMR}\mathsf{el}\mathcal{C}\mathsf{at}_\infty^{(2)}$ and $\mathcal{SMC}\mathsf{at}_\infty^{(2)}$. For each $(\mathcal{C}^\otimes, \mathcal{W}_\mathcal{C}) \in \mathcal{X}$, let $\mathcal{C}^{\otimes,\mathcal{W}_\mathcal{C}}$ denote the marked simplicial set obtained from $\mathcal{C}^\otimes$ by marking the cocartesian edges and the equivalences in $\mathcal{W}_\mathcal{C}$. The assignment $(\mathcal{C}^\otimes, \mathcal{W}_\mathcal{C}) \mapsto \mathcal{C}^{\otimes,\mathcal{W}_\mathcal{C}}$ determines an enriched functor

$$\mathcal{X} \to \mathsf{sSet}^+_{/\mathsf{Fin}_*}.$$

In general, given a pair of objects $(\mathcal{C}^\otimes, \mathcal{W}_\mathcal{C}), (\mathcal{D}^\otimes, \mathcal{W}_\mathcal{D}) \in \mathcal{X}$ the map

$$\mathcal{X}\left((\mathcal{C}^\otimes, \mathcal{W}_\mathcal{C}), (\mathcal{D}^\otimes, \mathcal{W}_\mathcal{D})\right) \to [\mathcal{C}^{\otimes,\mathcal{W}_\mathcal{C}}, \mathcal{D}^{\otimes,\mathcal{W}_\mathcal{D}}]$$

is *not* an isomorphism, but it is an isomorphism when $(\mathcal{D}^\otimes, \mathcal{W}_\mathcal{D}) \in \mathcal{Y}$ (i.e., $\mathcal{W}_\mathcal{D}$ are the equivalences of $\mathcal{D}$). Therefore, L_κ restricts to a functor $L\colon \mathcal{X} \to \mathcal{Y}$, and η restricts to an enriched natural transformation $\eta\colon \mathrm{id}_\mathcal{X} \Rightarrow L$. The desired functor and natural transformation is given by taking the scaled nerve of L and η. ☐

Proof of Theorem 4.1. By Remark 4.5, we may replace $\mathsf{TractSMMC}$ by $\mathsf{TractSMMC}_\emptyset$. We will first construct a natural transformation $\theta\colon \Sigma\mathrm{Seq}\,(-)^\circ \to \Sigma\mathrm{Seq}_B\,((-)_\infty)^\circ$ of functors $\mathsf{TractSMMC}_\emptyset \to \widehat{\mathcal{M}\mathsf{on}\mathcal{C}\mathsf{at}}_\infty$. Define a functor $\tau\colon \mathsf{TractSMMC}_\emptyset \to \int^{\mathsf{TractSMMC}_\emptyset} B\Sigma\mathrm{Seq}\,(-)^\circ$ by

$$\tau(\mathbf{M}) = (i_\mathbf{M}\colon \mathbf{M} \hookrightarrow \mathbf{M}[\mathcal{X}]).$$

By Proposition 4.9, τ is a cocartesian section of the projection $\int^{\mathsf{TractSMMC}_\emptyset} B\Sigma\mathrm{Seq}\,(-)^\circ \to \mathsf{TractSMMC}_\emptyset$. Postcompositon with the functor $(-)_\infty^\otimes : \mathsf{TractSMMC}_\emptyset^{(2)} \to \mathcal{P}\mathrm{r}\mathcal{SM}^{(2)}$ determines a functor

$$F\colon \int^{\mathsf{TractSMMC}_\emptyset} B\Sigma\mathrm{Seq}\,(-)^\circ \to \int^{\mathcal{P}\mathrm{r}\mathcal{SM}} \mathrm{CAlg}_\bullet$$

Part (3) of Proposition 3.2 shows that F preserves cocartesian edges. It follows from that the diagram

$$\begin{tikzcd}
\int^{\mathsf{TractSMMC}_\emptyset} B\Sigma\mathrm{Seq}(-)^\circ \arrow[r, "F"] & \int^{\mathcal{P}\mathsf{r}\mathcal{S}\mathcal{M}} \mathrm{CAlg}^{(2)}_\bullet \\
\mathsf{TractSMMC}_\emptyset \arrow[u, "\tau"] \arrow[r, "(-)^\otimes_\infty"'] & \mathcal{P}\mathsf{r}\mathcal{S}\mathcal{M} \arrow[u, "\sigma"']
\end{tikzcd}$$

commutes up to natural equivalence, where σ is the section defined in Remark 1.11. By straightening, this gives rise to a natural transformation

$$\theta\colon \Sigma\mathrm{Seq}\left(-\right)^\circ \to \Sigma\mathrm{Seq}_B\left(\left(-\right)_\infty\right)^\circ.$$

To complete the proof, we must show that for each $\mathbf{M} \in \mathsf{TractSMMC}_\emptyset$, the map

$$\theta_{\mathbf{M}}\colon \Sigma\mathrm{Seq}\left(\mathbf{M}\right) \to \Sigma\mathrm{Seq}_B\left(\mathbf{M}_\infty\right)$$

is a localization at weak equivalences. (See Remark 2.3.) For this, we will construct the functor $(-)^\otimes_\infty$ by using Lemma 4.10; this will ensure that $\mathbf{M}^\otimes_\infty \to \mathrm{Fun}\left(\mathsf{FB}, \mathbf{M}\right)^\otimes_\infty$ is a monomorphism of simplicial sets for all $\mathbf{M} \in \mathsf{TractSMMC}_\emptyset$. By part (2) of Proposition A.16, we can identify $\theta_{\mathbf{M}}$ with the dashed arrow in the diagram

$$\begin{tikzcd}
 & \mathrm{Fun}^{\otimes,L}_{\mathbf{M}_\infty/}(\mathbf{M}[\mathfrak{X}]_\infty, \mathbf{M}[\mathfrak{X}]_\infty) \arrow[rr] \arrow[dd] & & \{(i_{\mathbf{M}})^\otimes_\infty\} \arrow[dd] \\
\mathrm{Fun}^{\otimes,LQ}_{\mathbf{M}/}(\mathbf{M}[\mathfrak{X}], \mathbf{M}[\mathfrak{X}]) \arrow[ur, dashed, "\theta_{\mathbf{M}}"] \arrow[rr] \arrow[dd] & & \{i_{\mathbf{M}}\} \arrow[ur] \arrow[dd] & \\
 & \mathrm{Fun}^{\otimes,L}(\mathbf{M}[\mathfrak{X}]_\infty, \mathbf{M}[\mathfrak{X}]_\infty) \arrow[rr] & & \mathrm{Fun}^{\otimes,L}(\mathbf{M}_\infty, \mathbf{M}[\mathfrak{X}]_\infty) \\
\mathrm{Fun}^{\otimes,LQ}(\mathbf{M}[\mathfrak{X}], \mathbf{M}[\mathfrak{X}]) \arrow[ur] \arrow[rr] & & \mathrm{Fun}^{\otimes,LQ}(\mathbf{M}, \mathbf{M}[\mathfrak{X}]), \arrow[ur] &
\end{tikzcd}$$

where the front and back faces are defined by strict pullback (but which happen to be homotopy pullback), and the slanted arrows of the bottom face is induced by the functor sSet^+-enriched functor $(-)^\otimes_\infty$. We now consider the following diagram:

$$\begin{tikzcd}
\mathrm{Fun}^{\otimes,LQ}_{\mathbf{M}/}(\mathbf{M}[\mathfrak{X}], \mathbf{M}[\mathfrak{X}]) \arrow[r, dashed, "\theta_{\mathbf{M}}"] \arrow[d] & \mathrm{Fun}^{\otimes,L}_{\mathbf{M}_\infty/}(\mathbf{M}[\mathfrak{X}]_\infty, \mathbf{M}[\mathfrak{X}]_\infty) \arrow[d] \\
\mathrm{Fun}^{\otimes,LQ}(\mathbf{M}[\mathfrak{X}], \mathbf{M}[\mathfrak{X}]) \arrow[r, "(-)^\otimes_\infty"] \arrow[d] & \mathrm{Fun}^{\otimes,L}(\mathbf{M}[\mathfrak{X}]_\infty, \mathbf{M}[\mathfrak{X}]_\infty) \arrow[d] \\
\mathrm{Fun}^{\otimes,\mathrm{rel}}(\mathbf{M}[\mathfrak{X}]_{\mathrm{cof}}, \mathbf{M}[\mathfrak{X}]_{\mathrm{cof}}) \arrow[r, "L"] \arrow[d] & \mathrm{Fun}^{\otimes}(\mathbf{M}[\mathfrak{X}]_\infty, \mathbf{M}[\mathfrak{X}]_\infty) \arrow[d] \\
\mathrm{Fun}^{\otimes,\mathrm{rel}}(\mathsf{FB}, \mathbf{M}[\mathfrak{X}]_{\mathrm{cof}}) \arrow[r, "L"] \arrow[dr, "(\eta_{\mathrm{Fun}(\mathsf{FB},\mathbf{M})_{\mathrm{cof}}})_*"'] \arrow[dd, "\epsilon"'] & \mathrm{Fun}^{\otimes}(L(\mathsf{FB}), \mathbf{M}[\mathfrak{X}]_\infty) \arrow[d, "\eta^*_{\mathsf{FB}}"] \\
 & \mathrm{Fun}^{\otimes}(\mathsf{FB}, \mathbf{M}[\mathfrak{X}]_\infty) \arrow[d, "\epsilon"] \\
\mathrm{Fun}(\mathsf{FB}, \mathbf{M})_{\mathrm{cof}} \arrow[r, "\eta_{\mathrm{Fun}(\mathsf{FB},\mathbf{M})_{\mathrm{cof}}}"'] & \mathrm{Fun}(\mathsf{FB}, \mathbf{M})_\infty.
\end{tikzcd}$$

Here FB is regarded as a symmetric monoidal relative ∞-category whose weak equivalences are exactly the isomorphisms, the symmetric monoidal functor $\mathsf{FB} \to \mathrm{Fun}\left(\mathsf{FB}, \mathbf{M}\right)$ carries the singleton $\underline{1}$ to the unit symmetric sequence $(\emptyset, \mathbf{1}, \emptyset, \emptyset, \dots)$, and ϵ is the evaluation at the singleton. The triangle commutes by the enriched naturality of η, and the remaining squares commute trivially. The vertical composites are equivalences by Propositions 1.12 and 4.7. Therefore, we are reduced to showing that the bottom horizontal arrow is a localization, which it is by construction. The proof is now complete. $\square$

5. Closer look at Haugseng's model

In this section, we prove the following analog of Theorem 4.1 for Haugseng's composition product monoidal structure:

Theorem 5.1. *There is a natural equivalence*

$$\Sigma\mathrm{Seq}\left(\mathbf{M}\right)_{\mathrm{cof}}^{\circ}\left[\mathrm{weq}^{-1}\right] \simeq \Sigma\mathrm{Seq}_{\mathrm{H}}\left(\mathbf{M}_{\infty}\right)^{\circ}$$

of functors $\mathsf{TractSMMC} \to \mathcal{M}\mathsf{on}\widehat{\mathcal{C}\mathsf{at}}_{\infty}$.

The remainder of this section is devoted to the proof of Theorem 5.1.

Our proof will rely on an alternative construction of Haugseng's model, which can take arbitrary symmetric monoidal ∞-categories as its input and has ∞-operads as its output. (In the 1-categorical setting, Ching developed a closely related idea was developed in [Chi12].) To describe the construction, we need to introduce a bit of notation.

Notation 5.2.

- We write $\Sigma \subset \mathrm{Fun}\left([1], \mathbf{\Delta}^{\mathrm{op}}\right)$ for the full subcategory spanned by the inert maps. The evaluation at $0 \in [1]$ determines a cartesian fibration $\mathrm{ev}_0\colon \Sigma \to \mathbf{\Delta}^{\mathrm{op}}$, whose fiber over $[n] \in \mathbf{\Delta}^{\mathrm{op}}$ can be identified with the poset of subintervals of $[n]$ ordered by reverse inclusion.
- We then define a category $\Sigma\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}}$ and functors $\pi_{\mathbf{\Delta}^{\mathrm{op}}}, \pi_{\mathsf{Fin}_*}$ by the commutative diagram

$$\begin{tikzcd}
\Sigma\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, bend left, "\pi_{\mathbf{\Delta}^{\mathrm{op}}}"] \arrow[dd, bend right=60, "\pi_{\mathsf{Fin}_*}"'] & \Sigma \arrow[r, "\mathrm{ev}_0"] \arrow[d, "\mathrm{ev}_1"] & \mathbf{\Delta}^{\mathrm{op}} \\
\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}} \arrow[r] \arrow[d, "V"] & \mathbf{\Delta}^{\mathrm{op}} & \\
\mathsf{Fin}_* & &
\end{tikzcd}$$

  whose top square is a pullback. We will think of $\Sigma\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}}$ as lying over Fin_* and $\mathbf{\Delta}^{\mathrm{op}}$ by these maps.

We typically denote an object $\iota\colon [n] \to [m]$ of Σ by $[n] \to [a, b]$, where $a = \iota(0)$ and $b = \iota(m)$. With this notation, a typical object of $\Sigma\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}}$ can be written as a pair

$$\left([n] \to [a, b], S_a \to \cdots \to S_b\right)$$

where $S_a \to \cdots \to S_b$ is a sequence of morphsims in Fin.

Remark 5.3. The functor $\pi_{\mathbf{\Delta}^{\mathrm{op}}}$ is a flat categorical fibration, since it is the composite of a cocartesian fibration and a cartesian fibration.

Notation 5.4. Given a simplicial set $X \in \mathsf{sSet}_{/\mathbf{\Delta}^{\mathrm{op}}}$ and a map $f\colon K \to \mathbf{\Delta}^{\mathrm{op}}$ of simplicial sets, we write $X_f = X_K = K \times_{\mathbf{\Delta}^{\mathrm{op}}} X$.

Definition 5.5. Let $p\colon \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ be an ∞-operad. We define a functor $\widetilde{\Sigma\mathrm{Seq}_{\mathrm{H}}}(\mathcal{O})^{\circ} \to \mathbf{\Delta}^{\mathrm{op}}$ to be the image of $\mathcal{O}^{\otimes} \in \mathsf{sSet}_{/\mathsf{Fin}_*}$ under the composite

$$\mathsf{sSet}_{/\mathsf{Fin}_*} \xrightarrow{(\pi_{\mathsf{Fin}_*})^*} \mathsf{sSet}_{/\Sigma\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}}} \xrightarrow{(\pi_{\mathbf{\Delta}^{\mathrm{op}}})_*} \mathsf{sSet}_{/\mathbf{\Delta}^{\mathrm{op}}}.$$

In other words, $\widetilde{\Sigma\mathrm{Seq}_{\mathrm{H}}}(\mathcal{O})^{\circ}$ is characterized by the universal property

$$\mathrm{Fun}_{\mathbf{\Delta}^{\mathrm{op}}}\left(K, \widetilde{\Sigma\mathrm{Seq}_{\mathrm{H}}}(\mathcal{O})^{\circ}\right) \cong \mathrm{Fun}_{\mathsf{Fin}_*}\left(\left(\Sigma\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}}\right)_K, \mathcal{O}^{\otimes}\right).$$

Note that the projection $\widetilde{\Sigma\mathrm{Seq}_{\mathrm{H}}}(\mathcal{O})^{\circ} \to \mathbf{\Delta}^{\mathrm{op}}$ is a categorical fibration by Remark 5.3. We write $\Sigma\mathrm{Seq}_{\mathrm{H}}(\mathcal{O})^{\circ} \subset \widetilde{\Sigma\mathrm{Seq}_{\mathrm{H}}}(\mathcal{O})^{\circ}$ for the full subcategory spanned by the functors $\{[n]\} \times_{\mathbf{\Delta}^{\mathrm{op}}} \Sigma\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}} \to \mathcal{O}^{\otimes}$ carrying every morphism to an inert map.

Remark 5.6. In the situation of Definition 5.5, the functor

$$\mathsf{FB} \to \{[1]\} \times_{\mathbf{\Delta}^{\mathrm{op}}} \Sigma\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}},\ S \mapsto (S \to \underline{1})$$

induces a categorical equivalence

$$\Sigma\mathrm{Seq}_{\mathrm{H}}(\mathcal{O}) \xrightarrow{\simeq} \mathrm{Fun}(\mathsf{FB}, \mathcal{O}).$$

Indeed, we can identify $\Sigma\mathrm{Seq}(\mathcal{O})$ with the full subcategory of $\mathrm{Fun}_{\mathsf{Fin}_*}(\{[1]\} \times_{\mathbf{\Delta}^{\mathrm{op}}} \Sigma\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}}, \mathcal{O}^{\otimes})$ spanned by the functors that are p-right Kan extended from FB.

The next two propositions assert that $\Sigma\mathrm{Seq}_{\mathrm{H}}(\mathcal{O})^{\circ}$ is an ∞-operad characterized by a certain universal property, and that the notation $\Sigma\mathrm{Seq}_{\mathrm{H}}(-)^{\circ}$ does not conflict with Theorem 1.2.

Proposition 5.7. *Let $p\colon \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ be an ∞-operad. The projection $q\colon \Sigma\mathrm{Seq}_{\mathrm{H}}(\mathcal{O})^{\circ} \to \mathbf{\Delta}^{\mathrm{op}}$ is a non-symmetric ∞-operad. Moreover, if $\alpha\colon X \to Y$ is a morphism of $\Sigma\mathrm{Seq}_{\mathrm{H}}(\mathcal{O})^{\circ}$ lying over an inert map $\underline{\alpha}\colon [n] \to [m]$, the following conditions are equivalent:*

- *(a) The morphism α is q-cocartesian.*
- *(b) The functor $(\Sigma\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}})_{\underline{\alpha}} \to \mathcal{O}^{\otimes}$ adjoint to α carries every morphism lying over an inert map of Fin_* to an inert map of $\mathcal{O}^{\otimes}$.*

Proposition 5.8. *Let $\mathcal{O}^{\otimes}$ be an ∞-operad, and let $\mathcal{P}^{\otimes}$ be a non-symmetric ∞-operad. The restriction along the diagonal map $\mathbf{\Delta}^{\mathrm{op}} \to \Sigma$ induces a categorical equivalence*

$$\mathrm{Alg}_{\mathcal{P}}(\Sigma\mathrm{Seq}_{\mathrm{H}}(\mathcal{O})) \xrightarrow{\simeq} \mathrm{Alg}^{\mathrm{opd}}_{\mathcal{P}^{\otimes} \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}}}(\mathcal{O}).$$

Proof of Proposition 5.7. It is tempting to use [Lur17, Theorem B.4.2], but condition (5) of loc. cit. is not satisfied. So we must take a bit of a detour. We will say that a morphism α of $\Sigma\mathrm{Seq}(\mathcal{O})^{\circ}$ that lies over an inert map is *special* if it satisfies condition (b) in the statement. We must prove the following:

1. For each inert map $\underline{\alpha} : [n] \to [m]$ of $\mathbf{\Delta}^{\mathrm{op}}$ and object $X \in \Sigma\mathrm{Seq}(\mathcal{O})^{\circ}_{[n]}$, there is a special morphism $\alpha : X \to Y$ lying over $\underline{\alpha}$.
2. Every special morphism of $\Sigma\mathrm{Seq}(\mathcal{O})^{\circ}$ is q-cocartesian.
3. For each $n \geq 0$ and each collection of objects $X_1, \dots, X_n \in \Sigma\mathrm{Seq}(\mathcal{O})$, there is an object $X \in \Sigma\mathrm{Seq}(\mathcal{O})$ that admits special maps $\{X \to X_i\}_{1 \leq i \leq n}$ lying over the inert maps $\{[n] \to [i-1, i] \cong [1]\}_{1 \leq i \leq n}$.
4. Any collection of maps $\{X \to X_i\}_{1 \leq i \leq n}$ as in (3) determines a q-limit diagram in $\Sigma\mathrm{Seq}(\mathcal{O})$.

We start with (1). There is a retraction $\rho\colon (\Sigma\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}})_{\underline{\alpha}} \to (\Sigma\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}})_{[n]}$ carrying each object $(\iota\colon [m] \to [i,j],\ S_i \to \cdots \to S_j) \in (\Sigma\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}})_{[m]}$ to $(\iota\underline{\alpha}\colon [n] \to [i,j],\ S_i \to \cdots \to S_j) \in (\Sigma\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}})_{[n]}$. Precomposing this retraction to the map $(\Sigma\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}})_{[n]} \to \mathcal{O}^{\otimes}$ adjoint to X, we get the desired special morphism.

Next, we prove (2). Suppose we are given a special morphism α of $\Sigma\mathrm{Seq}\,(\mathcal{O})^{\circ}$ described as in (b). We wish to show that α is p-cocartesian. Using [Lur17, Proposition B.4.9] to the functors $\mathcal{O}^{\otimes} \times_{\mathsf{Fin}_*} \Sigma\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}} \to \Sigma\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}} \xrightarrow{\mathrm{ev}_0} \mathbf{\Delta}^{\mathrm{op}}$ and the full subcategory $\emptyset \subset \Sigma\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}}$, we are reduced to showing that the functor

$$F\colon \left(\Sigma\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}}\right)_{\alpha} \to \mathcal{O}^{\otimes}$$

adjoint to α is p-left Kan extended from $\left(\Sigma\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}}\right)_0$. Since the projection $\pi\colon \left(\Sigma\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}}\right)_{\alpha} \to [1]$ is a cartesian fibration, this is equivalent to the condition that F carries π-cartesian morphisms to inert maps. But π-cartesian morphisms lies over isomorphisms of Fin_*, so the claim follows from the specialty of F.

We next turn to (3). Consider the discrete set $\underline{n} = \{1, \dots, n\}$ and the poset $\underline{n}^{\triangleleft}$ obtained from $\underline{n}$ by adjoining a minimal element $-\infty$. We let $\underline{n}^{\triangleleft} \to \mathbf{\Delta}^{\mathrm{op}}$ denote the functor carrying the map $-\infty \to i$ to the inert map $[n] \to [i-1, i] \cong [1]$. We can identify the objects $X_1, \dots, X_n$ with a functor $F\colon \left(\Sigma\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}}\right)_{\underline{n}} \to \mathcal{O}^{\otimes}$, and we must find a filler as indicated below

$$\begin{tikzcd}
(\Sigma\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}})_{\underline{n}} \arrow[r, "F"] \arrow[d, hook] & \mathcal{O}^{\otimes} \arrow[d, "p"] \\
(\Sigma\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}})_{\underline{n}^{\triangleleft}} \arrow[r] \arrow[ru, dashed] & \mathsf{Fin}_*,
\end{tikzcd}$$

which determines special maps of $\Sigma\mathrm{Seq}\,(\mathcal{O})^{\circ}$. We will construct the filler as a p-right Kan extension of F, and then show that this corresponds to n special maps of $\Sigma\mathrm{Seq}\,(\mathcal{O})^{\circ}$.

Consider an object $\xi = ([n] \to [i,j], S_i \to \cdots \to S_j) \in \left(\Sigma\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}}\right)_{[n]}$. We will assume that $i < j$, because the argument for the case where $i = j$ is similar. The inert maps $\{[i,j] \to [k-1,k] \cap [i,j]\}_{i \le k \le j+1}$ and the identity maps of the S_k's determine an initial map

$$\{i, \dots, j+1\} \hookrightarrow \left(\left(\Sigma\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}}\right)_{\underline{n}}\right)_{\xi/}.$$

Therefore, to prove the existence of a p-right Kan extension, it will suffice to show that there is a p-limit diagram $\{i, \dots, j+1\}^{\triangleleft} \to \mathcal{O}^{\otimes}$ lying over the composite $\{i, \dots, j+1\}^{\triangleleft} \to \left(\Sigma\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}}\right)_{\underline{n}^{\triangleleft}} \to \mathsf{Fin}_*$. This follows from the definition of ∞-operads. Moreover, this argument shows that the p-right Kan extension $\overline{F} : \underline{n}^{\triangleleft} \times_{\mathbf{\Delta}^{\mathrm{op}}} \Sigma\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}} \to \mathcal{O}^{\otimes}$ determines special maps of $\Sigma\mathrm{Seq}\,(\mathcal{O})^{\circ}$.

The proof of part (4) is similar to that of part (3), using [Lur17, Proposition B.4.9] again. The proof is now complete. □

To facilitate the proof of Proposition 5.8, we will use *marked simplicial sets*, which is a useful gadget to keep track of a designated class of morphisms (such as inert maps).

Recollection 5.9. A **marked simplicial set** is a pair (S, M), where S is a simplicial set and M is a set of edges of S containing all degenerate edges. If (X, E) and (X', E') are marked simplicial sets equipped with maps $p\colon X \to S$ and $p'\colon X' \to S$ of simplicial sets, we write $\mathrm{Fun}_S\left((X, E), (X', E')\right)$ for the full simplicial subset of $\mathrm{Fun}_S\left(X, X'\right) = \mathrm{Fun}\left(X, X'\right) \times_{\mathrm{Fun}(X', S)} \{p'\}$ spanned by the maps $X \to X'$ carrying E into E'.

If $\mathcal{O}^{\otimes}$ is a symmetric or non-symmetric ∞-operad, we write $\mathcal{O}^{\otimes,\natural}$ for the marked simplicial set obtained from $\mathcal{O}^{\otimes}$ by marking the inert edges. If X is a simplicial set, we write $X^{\sharp}$ for the marked simplicial set obtained from X by marking *all* edges.

Proof of Proposition 5.8. Write $\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op},\natural}$ for the marked simplicial set obtained from $\mathbf{\Delta}^{\mathrm{op}}$ by marking the edges lying over inert maps of $\mathbf{\Delta}^{\mathrm{op}}$. By Proposition 5.7, the marked simplicial set $\Sigma\mathrm{Seq}\left(\mathcal{O}\right)^{\circ,\natural} \in \mathsf{sSet}^{+}_{/\boldsymbol{\Delta}^{\mathrm{op},\natural}}$ is characterized by the isomorphism of simplicial sets

$$\mathrm{Fun}_{\boldsymbol{\Delta}^{\mathrm{op}}}\left(\overline{K}, \Sigma\mathrm{Seq}_{\mathrm{H}}\left(\mathcal{O}\right)^{\natural}\right) \cong \mathrm{Fun}_{\mathsf{Fin}_*}\left(\overline{K} \times_{\left(\boldsymbol{\Delta}^{\mathrm{op},\natural}\right)^{\{0\}^\sharp}} \left(\boldsymbol{\Delta}^{\mathrm{op},\natural}\right)^{[1]^\sharp} \times_{\left(\boldsymbol{\Delta}^{\mathrm{op},\natural}\right)^{\{1\}^\sharp}} \boldsymbol{\Delta}_{\mathbb{F}}^{\mathrm{op},\natural}, \mathcal{O}^{\otimes,\natural}\right)$$

natural in $\overline{K} \in \mathsf{sSet}^{+}_{/\boldsymbol{\Delta}^{\mathrm{op},\natural}}$. In light of this, it suffices to show that the map

$$\theta\colon \mathcal{P}^{\otimes,\natural} \times_{\boldsymbol{\Delta}^{\mathrm{op},\natural}} \boldsymbol{\Delta}_{\mathbb{F}}^{\mathrm{op},\natural} \to \mathcal{P}^{\otimes,\natural} \times_{\left(\boldsymbol{\Delta}^{\mathrm{op},\natural}\right)^{\{0\}^\sharp}} \left(\boldsymbol{\Delta}^{\mathrm{op},\natural}\right)^{[1]^\sharp} \times_{\left(\boldsymbol{\Delta}^{\mathrm{op},\natural}\right)^{\{1\}^\sharp}} \boldsymbol{\Delta}_{\mathbb{F}}^{\mathrm{op},\natural}$$

is a weak equivalence in the model category of ∞-preoperads [Lur17, Proposition 2.1.4.6].

We can factor θ as

$$\begin{aligned} \mathcal{P}^{\otimes,\natural} \times_{\boldsymbol{\Delta}^{\mathrm{op},\natural}} \boldsymbol{\Delta}_{\mathbb{F}}^{\mathrm{op},\natural} &\xrightarrow{\theta'} \left(\mathcal{P}^{\otimes,\natural}\right)^{[1]^\sharp} \times_{\left(\boldsymbol{\Delta}^{\mathrm{op},\natural}\right)^{\{1\}^\sharp}} \boldsymbol{\Delta}_{\mathbb{F}}^{\mathrm{op},\natural} \\ &\xrightarrow{\theta''} \mathcal{P}^{\otimes,\natural} \times_{\left(\boldsymbol{\Delta}^{\mathrm{op},\natural}\right)^{\{0\}^\sharp}} \left(\boldsymbol{\Delta}^{\mathrm{op},\natural}\right)^{[1]^\sharp} \times_{\left(\boldsymbol{\Delta}^{\mathrm{op},\natural}\right)^{\{1\}^\sharp}} \boldsymbol{\Delta}_{\mathbb{F}}^{\mathrm{op},\natural}. \end{aligned}$$

The map θ' is a weak equivalence because it has a homotopy inverse, given by the evaluation at $1 \in [1]$. The second map is a pullback of the map $\left(\mathcal{P}^{\otimes,\natural}\right)^{[1]^\sharp} \to \mathcal{P}^{\otimes,\natural} \times_{\left(\boldsymbol{\Delta}^{\mathrm{op},\natural}\right)^{\{0\}^\sharp}} \left(\boldsymbol{\Delta}^{\mathrm{op},\natural}\right)^{[1]^\sharp}$, which is a trivial fibration by [Lur17, Proposition B.1.9]. Hence θ is a weak equivalence, as claimed. ☐

We now focus on the ∞-operad $\Sigma\mathrm{Seq}_{\mathrm{H}}\left(\mathcal{C}\right)^{\circ}$ in the case where $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category. We will show that it is a monoidal ∞-category when $\mathcal{C}^{\otimes}$ is compatible with colimits indexed by countable groupoids (Corollary 5.17), and that it is compatible with the classical composition product (Proposition 5.18).

Construction 5.10. Every ∞-operad $\mathcal{O}^{\otimes}$ can be freely made into a symmetric monoidal ∞-category $\mathrm{Env}\left(\mathcal{O}\right)^{\otimes}$, called the **symmetric monoidal envelop** [Lur17, Construction 2.2.4.1, Proposition 2.2.4.9]. Concretely, $\mathrm{Env}\left(\mathcal{O}\right)^{\otimes}$ is given by the fiber product

$$\begin{tikzcd} \mathrm{Env}(\mathcal{O})^{\otimes} \arrow[r, "p_{\mathcal{O}}"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{O}^{\otimes} \arrow[d] \\ \mathrm{Fun}^{\mathrm{act}}([1], \mathsf{Fin}_*) \arrow[r, "\mathrm{ev}_0"'] & \mathsf{Fin}_* \end{tikzcd}$$

with structure map $\mathrm{Env}\left(\mathcal{O}\right)^{\otimes} \to \mathsf{Fin}_*$ given by the evaluation at $1 \in [1]$, where $\mathrm{Fun}^{\mathrm{act}}\left([1], \mathsf{Fin}_*\right) \subset \mathrm{Fun}\left([1], \mathsf{Fin}_*\right)$ denotes the full subcategory spanned by the active maps. The diagonal $\mathsf{Fin}_* \to \mathrm{Fun}^{\mathrm{act}}\left([1], \mathsf{Fin}_*\right)$ determines a map of ∞-operads $\eta\colon \mathcal{O}^{\otimes} \to \mathrm{Env}\left(\mathcal{O}\right)^{\otimes}$, and for every symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$, pulling back along η induces a categorical equivalence

$$\mathrm{Fun}^{\otimes}\left(\mathrm{Env}\left(\mathcal{O}\right), \mathcal{C}\right) \xrightarrow{\simeq} \mathrm{Alg}_{\mathcal{O}}\left(\mathcal{C}\right).$$

Note that the underlying ∞-category $\mathrm{Env}\left(\mathcal{O}\right)$ can be identified with the subcategory $\mathcal{O}^{\otimes}_{\mathrm{act}} \subset \mathcal{O}^{\otimes}$ of active maps. We write $\oplus\colon \mathcal{O}^{\otimes}_{\mathrm{act}} \times \mathcal{O}^{\otimes}_{\mathrm{act}} \to \mathcal{O}^{\otimes}_{\mathrm{act}}$ for the tensor bifunctor of $\mathrm{Env}\left(\mathcal{O}\right)$.

Applying the above construction to $\mathcal{O}^{\otimes} = \mathcal{C}^{\otimes}$, we obtain an essentially unique symmetric monoidal functor $\mathrm{Env}\left(\mathcal{C}\right)^{\otimes} \to \mathcal{C}^{\otimes}$ that extends the identity of $\mathcal{C}^{\otimes}$ up to equivalence. (One way to construct it is to choose a cocartesian natural transformation rendering the diagram

$$\begin{tikzcd} \mathrm{Env}(\mathcal{C})^{\otimes} \times \{0\} \arrow[rr, "p_{\mathcal{C}}"] \arrow[d] & & \mathcal{C}^{\otimes} \arrow[d] \\ \mathrm{Env}(\mathcal{C})^{\otimes} \times [1] \arrow[r] \arrow[urr, dashed] & \mathrm{Fun}^{\mathrm{act}}([1], \mathsf{Fin}_*) \times [1] \arrow[r, "\mathrm{ev}"'] & \mathsf{Fin}_* \end{tikzcd}$$

commutative, and then restricting it along the inclusion $\{1\} \subset [1]$.) We denote its underlying functor by $\bigotimes\colon \mathcal{C}^{\otimes}_{\mathrm{act}} \to \mathcal{C}$.

Notation 5.11. We let $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{act}} \subset \mathbf{\Delta}^{\mathrm{op}}$ denote the subcategory of active morphisms. For each $n \geq 1$, we will denote the unique active map $[n] \to [1]$ by μ_n.

Notation 5.12. Let $\mathcal{F} \subset \mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}}$ denote the subcategory spanned by the morphisms $(\phi, \{u_i\}_i)\colon ([n], S_0 \to \cdots \to S_n) \to ([m], T_0 \to \cdots \to T_m)$ such that the maps u_i are all *bijective*. In other words, the map $\mathcal{F} \to \mathbf{\Delta}^{\mathrm{op}}$ is the cocartesian fibration corresponding to the functor

$$\mathbf{\Delta}^{\mathrm{op}} \to \mathsf{Cat},\ [n] \mapsto \left(\mathrm{Fun}\left([n], \mathsf{Fin}\right)^{\mathrm{op}}\right)^{\simeq}.$$

We write $\mathcal{G} \subset \mathcal{F}$ for the full subcategory spanned by the objects $([n], S_0 \to \cdots \to S_n)$ such that $S_n = \underline{1}$ is a singleton.

Construction 5.13. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. Suppose we are given a morphism f of $\Sigma\mathrm{Seq}_{\mathrm{H}}(\mathcal{C})^{\circ}$ lying over the active map $\mu_n\colon [n] \to [1]$, which we can identify with a functor $\left(\Sigma\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}}\right)_{\mu_n} \to \mathcal{C}^{\otimes}$. We define a natural transformation α_f of functors $\mathcal{F}_{[n]} \to \mathcal{C}$ as follows: Consider the composite

$$[1] \times \mathcal{F}_{[n]} \xrightarrow{h} \mathcal{F}_{\mu_n} \xrightarrow{g} \left(\Sigma\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}}\right)_{\mu_n} \xrightarrow{f} \mathcal{C}^{\otimes},$$

where h is a cocartesian natural transformation, and where g is induced by the diagonal map $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Fun}\left([1], \mathbf{\Delta}^{\mathrm{op}}\right)$. The above composite takes values in $\mathcal{C}^{\otimes}_{\mathrm{act}}$, so it can further be composed with the functor $\bigotimes\colon \mathcal{C}^{\otimes}_{\mathrm{act}} \to \mathcal{C}$ of Construction 5.10. We define $\alpha_f = \bigotimes \circ f \circ g \circ h$.

Remark 5.14. In the situation of Construction 5.13, write $f\colon X \to Y$ and $X = X_1 \oplus \cdots \oplus X_n$, where $X_i \in \Sigma\mathrm{Seq}_{\mathrm{H}}(\mathcal{C})$. By Remark 5.6, we can identify X_i and Y with symmetric sequences $\overline{X}_i$ and $\overline{Y}$ in $\mathcal{C}$. The component of the natural transformation α_f at an object $(S_0 \to \cdots \to S_n) \in \mathcal{F}_{[n]}$ is given by

$$\overline{X}_1\left(S_0 \to S_1\right) \otimes \cdots \otimes \overline{X}_n\left(S_{n-1} \to S_n\right) \to \overline{Y}\left(S_0 \to S_n\right),$$

where $\overline{Y}\left(A \to B\right)$ is a shorthand for $\bigotimes_{b \in B} \overline{Y}\left(A_b\right)$ and we used similar abbreviations for $\overline{X}_i\left(S_{i-1} \to S_i\right)$.

Proposition 5.15. *Let $p\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ be a symmetric monoidal ∞-category, and let $q\colon \Sigma\mathrm{Seq}(\mathcal{C})^{\circ} \to \mathbf{\Delta}^{\mathrm{op}}$ be the associated non-symmetric ∞-operad.*

1. *Let f be an active map of $\Sigma\mathrm{Seq}(\mathcal{C})^{\circ}$ lying over $\mu_n\colon [n] \to [1]$, which we identify with a functor $\left(\Sigma\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}}\right)_{\mu_n} \to \mathcal{C}$. Suppose that the following condition is satisfied:*
   - *($*_{\mathcal{G}}$) The natural transformation $\alpha_f\colon \mathcal{G}_{[n]} \times [1] \to \mathcal{C}$ exhibits $f|\mathcal{G}_{[1]}$ as a left Kan extension of $\alpha_f|\mathcal{G}_{[n]} \times \{0\}$ along the functor $\left(\mu_n\right)_!\colon \mathcal{G}_{[n]} \to \mathcal{G}_{[1]}$.*

   Then f is locally q-cocartesian.
2. *Let $X \in \Sigma\mathrm{Seq}(\mathcal{C})^{\circ}_{[n]}$. If the composite*

$$\theta_X\colon \mathcal{G}_{[n]} \to \left(\Sigma\Delta^{\mathrm{op}}_{\mathbb{F}}\right)_{[n]} \to \mathcal{C}^{\otimes}_{\mathrm{act}} \xrightarrow{\bigotimes} \mathcal{C}$$

   admits a left Kan extension along $\left(\mu_n\right)_!\,\theta_X\colon \mathcal{G}_{[n]} \to \mathcal{G}_{[1]}$, then there is a locally q-cocartesian morphism $X \to \left(\mu_n\right)_! X$ lying over μ_n.

The proof of Proposition 5.15 relies on the following lemma.

Lemma 5.16. *Let $p\colon \mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ be an ∞-operad. For each $n \geq 1$, the active map $[n] \to [1]$ determines a homotopy cartesian square of ∞-categories*

$$\begin{tikzcd}
\mathrm{Fun}_{\mathbf{\Delta}^{\mathrm{op}}}([1], \Sigma\mathrm{Seq}(\mathcal{O})^{\circ}) \arrow[r] \arrow[d] & \Sigma\mathrm{Seq}(\mathcal{O})^{\circ}_{[n]} \arrow[d] \\
\mathrm{Fun}'_{\mathsf{Fin}_*}((\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}})_{\mu_n}, \mathcal{O}^{\otimes}) \arrow[r] & \mathrm{Fun}'_{\mathsf{Fin}_*}((\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}})_{[n]}, \mathcal{O}^{\otimes}).
\end{tikzcd}$$

Here $\mathrm{Fun}'_{\mathsf{Fin}_*}(-,-)$ denotes the full subcategory of $\mathrm{Fun}_{\mathsf{Fin}_*}(-,-)$ spanned by the functors carrying each morphism in $(\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}})_{[n]}$ and $(\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}})_{[1]}$ to inert maps of $\mathcal{O}^{\otimes}$.

Proof. Set $\mathcal{X} = (\Sigma\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}})_{\mu_n}$ and $\mathcal{X}' = (\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}})_{\mu_n}$. We will identify $\mathcal{X}'$ with the full subcategory of $\mathcal{X}$ via the diagonal embedding $\mathbf{\Delta}^{\mathrm{op}} \to \Sigma$. For each $i \in [1]$, let M_i and M'_i denote the set of morphisms of $\mathcal{X}$ and $\mathcal{X}'$ lying over $i \in [1]$. Our goal is to show that the square

$$\begin{tikzcd}
\mathrm{Fun}_{\mathsf{Fin}_*}((\mathcal{X}, M_0 \cup M_1), \mathcal{O}^{\otimes,\natural}) \arrow[r] \arrow[d] & \mathrm{Fun}_{\mathsf{Fin}_*}((\mathcal{X}_0, M_0), \mathcal{O}^{\otimes,\natural}) \arrow[d] \\
\mathrm{Fun}_{\mathsf{Fin}_*}((\mathcal{X}', M'_0 \cup M'_1), \mathcal{O}^{\otimes,\natural}) \arrow[r] & \mathrm{Fun}_{\mathsf{Fin}_*}((\mathcal{X}'_0, M'_0), \mathcal{O}^{\otimes,\natural})
\end{tikzcd}$$

is homotopy cartesian.

Notice that if $X \to Y$ is a map of $\mathcal{X}$ such that $X \in \mathcal{X}_0$ and $Y \in \mathcal{X}'_1$, then X necessarily lies in $\mathcal{X}'_0$. It follows that $\mathcal{X}_0 \cup \mathcal{X}'$ is a full subcategory of $\mathcal{X}$. Moreover, for each $X \in \mathcal{X}$ lying outside of $\mathcal{X}_0 \cup \mathcal{X}'$, the category $(\mathcal{X}_0 \cup \mathcal{X}') \times_{\mathcal{X}} \mathcal{X}_{X/}$ is empty. Since every functor $F\colon \mathcal{X} \to \mathcal{O}^{\otimes}$ carries objects outside of $\mathcal{X}_0 \cup \mathcal{X}'$ to a p-terminal object (as they lies over $\langle 0 \rangle \in \mathsf{Fin}_*$), this means that the functor

$$\mathrm{Fun}_{\mathsf{Fin}_*}\left(\mathcal{X}, \mathcal{O}^{\otimes}\right) \to \mathrm{Fun}_{\mathsf{Fin}_*}\left(\mathcal{X}_0 \cup \mathcal{X}', \mathcal{O}^{\otimes}\right)$$

is an equivalence. In particular, the square

$$\begin{tikzcd}
\mathrm{Fun}_{\mathsf{Fin}_*}((\mathcal{X}, M_0 \cup M'_1), \mathcal{O}^{\otimes,\natural}) \arrow[r] \arrow[d] & \mathrm{Fun}_{\mathsf{Fin}_*}((\mathcal{X}_0, M_0), \mathcal{O}^{\otimes,\natural}) \arrow[d] \\
\mathrm{Fun}_{\mathsf{Fin}_*}((\mathcal{X}', M'_0 \cup M'_1), \mathcal{O}^{\otimes,\natural}) \arrow[r] & \mathrm{Fun}_{\mathsf{Fin}_*}((\mathcal{X}'_0, M'_0), \mathcal{O}^{\otimes,\natural})
\end{tikzcd}$$

is homotopy cartesian. The claim now follows from the observation that every functor $\mathcal{X} \to \mathcal{O}^{\otimes,\natural}$ carries every morphism in $M_1 \setminus M'_1$ to a p-cocartesian morphism (because its codomain lies over $\langle 0 \rangle \in \mathsf{Fin}_*$). ☐

Proof of Proposition 5.15. We start with (1). Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}_{\mathbf{\Delta}^{\mathrm{op}}}([1], \Sigma\mathrm{Seq}(\mathcal{C})^{\circ}) \arrow[r] \arrow[d] & \Sigma\mathrm{Seq}(\mathcal{C})^{\circ}_{[n]} \arrow[d] \\
\mathrm{Fun}'_{\mathsf{Fin}_*}((\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}})_{\mu_n}, \mathcal{C}^{\otimes}) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Fun}'_{\mathsf{Fin}_*}((\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}})_{[n]}, \mathcal{C}^{\otimes}) \arrow[d, "\simeq"] \\
\mathrm{Fun}_{\mathsf{Fin}_*}(\mathcal{G}_{\mu_n}, \mathcal{C}^{\otimes}) \arrow[r] & \mathrm{Fun}_{\mathsf{Fin}_*}(\mathcal{G}_{[n]}, \mathcal{C}^{\otimes}).
\end{tikzcd}$$

The top square is homotopy cartesian by Lemma 5.16. The right bottom vertical arrows is an equivalence because functors in $\mathrm{Fun}'_{\mathsf{Fin}_*}\left((\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}})_{[n]}, \mathcal{O}^{\otimes}\right)$ are exactly those that are p-right Kan extended from $\mathcal{G}_{[n]}$. Similarly, the left bottom vertical arrow is an equivalence. It follows that the outer square is homotopy cartesian. In particular, the map

$$\begin{aligned}&\mathrm{Fun}_{\mathbf{\Delta}^{\mathrm{op}}}\left([1],\Sigma\mathrm{Seq}\left(\mathcal{C}\right)^{\circ}\right)\times_{\Sigma\mathrm{Seq}(\mathcal{C})^{\circ}_{[n]}}\{f|\Sigma\mathrm{Seq}\left(\mathcal{C}\right)^{\circ}_{[n]}\}\\ \to{}&\mathrm{Fun}_{\mathsf{Fin}_*}\left(\mathcal{G}_{\mu_n},\mathcal{C}^{\otimes}\right)\times_{\mathrm{Fun}_{\mathsf{Fin}_*}\left(\mathcal{G}_{[n]},\mathcal{C}^{\otimes}\right)}\{f|\mathcal{G}_{[n]}\}\end{aligned}$$

is a categorical equivalence. Therefore, it suffices to show that $f|\mathcal{G}_{\mu_n}$ is a p-left Kan extension of $f|\mathcal{G}_{[n]}$. According to [Lur09b, Propositions 4.3.1.9, 4.3.1.10, and 4.3.1.15], this is equivalent to the condition that the composite

$$\mathcal{G}_{\mu_n}\xrightarrow{f|\mathcal{G}_{\mu_n}}\mathcal{C}^{\otimes}_{\mathrm{act}}\xrightarrow{\bigotimes}\mathcal{C}$$

be left Kan extended from $\mathcal{G}_{[n]}$. But this is true by $(*_{\mathcal{G}})$, and we are done.

For (2), note that the above argument shows that the image of X in $\mathrm{Fun}_{\mathsf{Fin}_*}\left(\mathcal{G}_{[n]},\mathcal{C}^{\otimes}\right)$ admits a p-left Kan extension. Since the outer rectangle of the diagram is homotopy cartesian, the p-left Kan extension gives rise to a morphism $f\colon X\to Y$ in $\Sigma\mathrm{Seq}\left(\mathcal{C}\right)^{\circ}$ satisfying condition $(*_{\mathcal{G}})$. The map f is locally q-cocartesian by (1), and this proves (2). □

Corollary 5.17. *Let $p\colon\mathcal{C}^{\otimes}\to\mathsf{Fin}_*$ be a symmetric monoidal ∞-category compatible with colimits indexed by countable groupoids. Then $q\colon\Sigma\mathrm{Seq}\left(\mathcal{C}\right)^{\circ}\to\mathbf{\Delta}^{\mathrm{op}}$ is a monoidal ∞-category. Moreover, an active map f of $\Sigma\mathrm{Seq}\left(\mathcal{C}\right)^{\circ}$ lying over $\mu_n\colon[n]\to[1]$ is q-cocartesian if and only if it satisfies the following condition:*

- $(*_{\mathcal{F}})$ *The natural transformation $\alpha_f\colon\mathcal{F}_{[n]}\times[1]\to\mathcal{C}$ exhibits $f|\mathcal{F}_{[1]}$ as a left Kan extension of $\alpha_f|\mathcal{F}_{[n]}\times\{0\}$ along the functor $\left(\mu_n\right)_!\colon\mathcal{F}_{[n]}\to\mathcal{F}_{[1]}$.*

Proof. We first show that an active map $f\colon X\to Y$ of $\Sigma\mathrm{Seq}\left(\mathcal{C}\right)^{\circ}$ lying over $\mu_n\colon[n]\to[1]$ is locally q-cocartesian if and only if it satisfies condition $(*_{\mathcal{F}})$. By Proposition 5.15, it suffices to show that condition $(*_{\mathcal{F}})$ is equivalent to condition $(*_{\mathcal{G}})$ there. Clearly $(*_{\mathcal{F}})$ implies $(*_{\mathcal{G}})$, so we only have to prove the converse.

Suppose that condition $(*_{\mathcal{G}})$ is satisfied. We must show that, for each $S\to T\in\mathcal{F}_{[1]}$, the map

(5.1)
$$\mathrm{colim}_{S_0\to\cdots\to S_n\in\mathcal{F}_{[n]/S\to T}}\bigotimes\circ X\left(\mathrm{id}_{[n]},S_0\to\cdots\to S_n\right)\to\bigotimes\circ Y\left(\mathrm{id}_{[1]},S\to T\right)$$

is an equivalence. More formally, we must show that the top horizontal composite in the diagram below is a colimit diagram:

$$\begin{tikzcd}
(\mathcal{F}_{[n]/S\to T})^{\triangleright} \arrow[r] \arrow[d, "\simeq"'] & \mathcal{F}_{\mu_n} \arrow[r] & (\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}})_{\mu_n} \arrow[r] & \mathcal{C}^{\otimes}_{\mathrm{act}} \arrow[r, "\otimes"] & \mathcal{C}. \\
(\prod_{t\in T}\mathcal{F}_{[n]/S\to\{t\}})^{\triangleright} \arrow[r] & \prod_{t\in T}((\mathcal{F}_{[n]/S\to\{t\}})^{\triangleright}) \arrow[r] & \prod_{t\in T}(\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}})_{\mu_n} \arrow[r] & \prod_{t\in T}\mathcal{C}^{\otimes}_{\mathrm{act}} \arrow[r, "\prod_{t\in T}\otimes"'] \arrow[u, "\oplus_{t\in T}"] & \prod_{t\in T}\mathcal{C} \arrow[u, "\otimes_{t\in T}"']
\end{tikzcd}$$

The left hand rectangle commutes by [Ara25, Corollary 4.5], and the right hand rectangle commutes because $\bigotimes\colon\mathcal{C}^{\otimes}_{\mathrm{act}}\to\mathcal{C}$ is symmetric monoidal. In other words, the map (5.1) can be identified with

$$\mathrm{colim}_{S_0\to\cdots\to S_n\in\mathcal{F}_{[n]/S\to T}}\bigotimes_{t\in T}\left(\bigotimes\circ X\left(\mathrm{id}_{[n]},S_{0,t}\to\cdots\to S_{n,t}\right)\xrightarrow{\simeq}\bigotimes\circ Y\left(\mathrm{id}_{[1]},S\to\{t\}\right)\right).$$

Since $\mathcal{C}^{\otimes}$ is compatible with colimits indexed by countable groupoids, we are therefore reduced to showing to that each of the maps $\{\left(\mathcal{F}_{[n]/S\to\{t\}}\right)^{\triangleright}\to\mathcal{C}\}_{t\in T}$ is a colimit diagram. This follows from $(*_{\mathcal{G}})$.

Next, we show that $\Sigma\mathrm{Seq}\left(\mathcal{C}\right)^{\circ}$ is a monoidal ∞-category. According to [Hau22, Lemma 2.1.25], the map q is a locally cocartesian fibration, and we only have to show that the composite of two locally q-cocartesian maps lying over active maps

$\phi\colon [n] \to [2]$ and $\mu_2\colon [2] \to [1]$ is again locally cocartesian. So take such maps $u\colon X \to Y$ and $v\colon Y \to Z$, and let $w\colon X \to Z$ denote a composite of v and u:

$$\begin{tikzcd}
& Y \arrow[dr, "v"] & & & {[2]} \arrow[dr, "\mu_2"] & \\
X \arrow[ur, "u"] \arrow[rr, "w"'] & & Z & {[n]} \arrow[ur, "\phi"] \arrow[rr, "\mu_n"'] & & {[1]}
\end{tikzcd}$$

We must show that w is also locally q-cocartesian. Using $(*_{\mathcal{F}})$ and the transitivity of Kan extensions, we are reduced to showing that the composite

$$\mathcal{F}_\phi \to \mathcal{C}^{\otimes}_{\mathrm{act}} \xrightarrow{\otimes} \mathcal{C}$$

is left Kan extended from $\mathcal{F}_{[n]}$. So take an arbitrary object $(T_0 \to T_1 \to T_2) \in \mathcal{F}_{[2]}$. We must show that the composite

$$\left(\mathcal{F}_{[n]/T_0 \to T_1 \to T_2}\right)^{\triangleright} \to \mathcal{F}_\phi \to \mathcal{C}^{\otimes}_{\mathrm{act}} \xrightarrow{\otimes} \mathcal{C}$$

is a colimit diagram. For this, find a diagram in $\Sigma\mathrm{Seq}\,(\mathcal{C})^{\circ}$ lying over the diagram on the right:

$$\begin{tikzcd}
X_1 \arrow[d, "u_1"'] & X \arrow[l, tail] \arrow[r, tail] \arrow[d, "u"'] & X_2 \arrow[d, "u_2"] & {[0,\phi(1)]} \arrow[d, "\phi_1"'] & {[n]} \arrow[l, tail] \arrow[r, tail] \arrow[d, "\phi"] & {[\phi(1),n]} \arrow[d, "\phi_2"] \\
Y_1 & Y \arrow[l, tail] \arrow[r, tail] & Y_2 & {[0,1]} & {[2]} \arrow[l, tail] \arrow[r, tail] & {[1,2]}
\end{tikzcd}$$

Here the arrows with tails "$\rightarrowtail$" are inert, and the remaining arrows are active. The maps u_1 and u_2 gives rise to maps $\mathcal{F}_{\mu_{\phi(1)}} \to \mathcal{C}^{\otimes}_{\mathrm{act}}$ and $\mathcal{F}_{\mu_{n-\phi(1)}} \to \mathcal{C}^{\otimes}_{\mathrm{act}}$, which fits into the following diagram:

$$\begin{tikzcd}
\left(\mathcal{F}_{[n]/T_0 \to T_1 \to T_2}\right)^{\triangleright} \arrow[rr] \arrow[d, "\simeq"'] & & \mathcal{F}_\phi \arrow[r] & \mathcal{C}^{\otimes}_{\mathrm{act}} \arrow[r, "\otimes"] & \mathcal{C} \\
\left(\mathcal{F}_{[0,\phi(1)]/T_0 \to T_1} \times \mathcal{F}_{[\phi(1),n]/T_1 \to T_2}\right)^{\triangleright} \arrow[r] & \left(\mathcal{F}_{[0,\phi(1)]/T_0 \to T_1}\right)^{\triangleright} \times \left(\mathcal{F}_{[\phi(1),n]/T_1 \to T_2}\right)^{\triangleright} \arrow[r] & \mathcal{F}_{\mu_{\phi(1)}} \times \mathcal{F}_{n-\phi(1)} \arrow[r] & \mathcal{C}^{\otimes}_{\mathrm{act}} \times \mathcal{C}^{\otimes}_{\mathrm{act}} \arrow[r, "\otimes \times \otimes"'] \arrow[u, "\oplus"] & \mathcal{C} \times \mathcal{C} \arrow[u, "\otimes"']
\end{tikzcd}$$

As before, the diagram commutes up to equivalence. Now u_1 and u_2 are locally q-cocartesian (see the argument of [Hau22, Lemma 2.1.25]), so by $(*_{\mathcal{F}})$, the maps

$$\left(\mathcal{F}_{[0,\phi(1)]/T_0 \to T_1}\right)^{\triangleright} \to \mathcal{C},\ \left(\mathcal{F}_{[\phi(1),n]/T_1 \to T_2}\right)^{\triangleright} \to \mathcal{C}$$

are colimit diagrams. Therefore, the claim follows from the compatibility of $\mathcal{C}^{\otimes}$ with colimits indexed by countable groupoids. The proof is now complete. ☐

Proposition 5.18. *Let $\mathcal{C}$ be a symmetric monoidal category compatible with countable groupoids. There is an equivalence of monoidal ∞-categories*

$$\Sigma\mathrm{Seq}\,(\mathcal{C})^{\circ} \xrightarrow{\simeq} \Sigma\mathrm{Seq}_{\mathrm{H}}\,(\mathcal{C})^{\circ}$$

which is natural with respect to monoidal functors preserving colimits indexed by countable groupoids.

Proof. We will construct a map $\Phi\colon \Sigma\mathrm{Seq}\,(\mathcal{C})^{\circ} \to \Sigma\mathrm{Seq}_{\mathrm{H}}\,(\mathcal{C})^{\circ}$ as follows: By construction, $\Sigma\mathrm{Seq}_{\mathrm{H}}\,(\mathcal{C})^{\circ}$ is isomorphic to the nerve of a category, so it suffices to specify Φ on objects and morphisms. An object of $\Sigma\mathrm{Seq}\,(\mathcal{C})^{\circ}$ is a finite sequence $(X_1, \dots, X_n)$ of symmetric sequences in $\mathcal{C}$. Its image $\Phi\,(X_1, \dots, X_n) \in \Sigma\mathrm{Seq}_{\mathrm{H}}\,(\mathcal{C})^{\circ}$ is given by the functor

$$\begin{gathered}
\left(\Sigma\mathbf{\Delta}^{\mathrm{op}}_{\mathbb{F}}\right)_{[n]} \to \mathcal{C}^{\otimes} \\
([n] \to [i,j], S_i \to \cdots \to S_j) \mapsto \left(X_p\,(S_{p-1,s})\right)_{(p,s) \in \coprod_{i<p\leq j} S_p}.
\end{gathered}$$

Next, given a morphism $f\colon \left(X_1,\dots,X_n\right)\to\left(Y_1,\dots,Y_m\right)$ in $\Sigma\mathrm{Seq}\left(\mathcal{C}\right)^{\circ}$ lying over a map $\underline{f}\colon [n]\to[m]$ in $\mathbf{\Delta}^{\mathrm{op}}$, the functor

$$\Phi\left(f\right):\ \left(\Sigma\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}}\right)_{\underline{f}}\to\mathcal{C}^{\otimes}$$

is defined as follows: The functor $\Phi\left(f\right)$ extends $\Phi\left(X_1,\dots,X_n\right)$ and $\Phi\left(Y_1,\dots,Y_m\right)$. If $\left([n]\to[i,j],S_i\to\cdots\to S_j\right)\to\left([m]\to[k,l],T_k\to\cdots\to T_l\right)$ is a morphism of $(\Sigma\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op}})_{\overline{f}}$ lying over the map $0\to 1$ in $[1]$, then its image $\left(X_p\left(S_{p-1,s}\right)\right)_{(p,s)\in\coprod_{i<p\leq j}S_p}\to\left(Y_q\left(T_{q-1,t}\right)\right)_{(q,t)\in\coprod_{k<q\leq l}T_q}$ is determined by the maps

$$\bigotimes_{\phi(q-1)<p\leq\phi(q)}\ \bigotimes_{s\in S_{p,t}}X_p\left(S_{p,s}\right)\to Y_q\left(S_{\phi(q-1),t}\right)\cong Y_q\left(T_{q-1,t}\right),$$

which in turn is determined by f. It is straightforward to check that Φ is a categorical equivalence and has the stated naturality. The claim follows. □

We now arrive at the proof of Theorem 5.1.

Proof of Theorem 5.1. Proposition 5.18 gives us a natural equivalence

$$\Sigma\mathrm{Seq}\left(\mathbf{M}\right)^{\circ}\xrightarrow{\simeq}\Sigma\mathrm{Seq}_{\mathrm{H}}\left(\mathbf{M}\right)^{\circ}.$$

Write $\Sigma\mathrm{Seq}_{\mathrm{H}}\left(\mathbf{M}\right)^{\circ}_{\mathrm{cof}}\subset\Sigma\mathrm{Seq}_{\mathrm{H}}\left(\mathbf{M}\right)^{\circ}$ for the essential image of $\Sigma\mathrm{Seq}\left(\mathbf{M}\right)^{\circ}_{\mathrm{cof}}$, which is a monoidal ∞-category. Since every projectively cofibrant symmetric sequence is injectively cofibrant (i.e., its values are cofibrant), we may consider the composite natural transformation

$$\theta_{\mathbf{M}}\colon\Sigma\mathrm{Seq}_{\mathrm{H}}\left(\mathbf{M}\right)^{\circ}_{\mathrm{cof}}\hookrightarrow\Sigma\mathrm{Seq}_{\mathrm{H}}\left(\mathbf{M}_{\mathrm{cof}}\right)^{\circ}\to\Sigma\mathrm{Seq}_{\mathrm{H}}\left(\mathbf{M}_{\infty}\right)^{\circ}.$$

A priori, the map $\theta_{\mathbf{M}}$ is merely lax monoidal, i.e., a map of non-symmetric ∞-operads. (In fact, $\Sigma\mathrm{Seq}_{\mathrm{H}}\left(\mathbf{M}_{\mathrm{cof}}\right)^{\circ}$ is generally not a monoidal ∞-category.) However, by Lemma 3.12 and Proposition 5.15, it is in fact *monoidal*.

To complete the proof, it suffices to show that $\theta_{\mathbf{M}}$ is a monoidal localization (Remark 2.3). By Remark 5.6, the underlying functor of $\theta_{\mathbf{M}}$ can be identified with the composite

$$\mathrm{Fun}\left(\mathsf{FB},\mathbf{M}\right)_{\mathrm{cof}}\xrightarrow{i}\mathrm{Fun}\left(\mathsf{FB},\mathbf{M}_{\mathrm{cof}}\right)\xrightarrow{j}\mathrm{Fun}\left(\mathsf{FB},\mathbf{M}_{\infty}\right).$$

The map i induces an equivalence upon localizing at weak equivalences, being the restriction of the Quillen equivalence between projective and injective model structures to the full subcategories of cofibrant objects. The map j is a localization by [Cis19, Theorem 7.9.8]. Hence $\theta_{\mathbf{M}}$ is a monoidal localization, and we are done. □

6. PROOF OF THE MAIN RESULT

We can finally give a proof of Theorem 1.17.

Proof of Theorem 1.17. By Theorem 2.5, it suffices to show that the functors

$$\Sigma\mathrm{Seq}_{\mathrm{B}}\left(\left(-\right)_{\infty}\right)^{\circ},\Sigma\mathrm{Seq}_{\mathrm{H}}\left(\left(-\right)_{\infty}\right)^{\circ}:\ \mathsf{TractSMMC}\to\mathcal{M}\mathrm{on}\widehat{\mathcal{C}\mathrm{at}}_{\infty}$$

are naturally equivalent. This follows from Theorems 4.1 and 5.1. □

APPENDIX A. RESULTS ON $(\infty,2)$-CATEGORIES

In this section, we summarize basic definitions and terminology related to $(\infty,2)$-categories that we will need in the paper.

A.1. **Definitions.** We will use ∞-bicategories as our preferred model of $(\infty, 2)$-categories. In this subsection, we recall the definitions and constructions related to this model.

Recollection A.1. [Lur09a, §3] A **scaled simplicial set** is a pair (X, T_X), where X is a simplicial set and T_X is a set of 2-simplices of X containing all the degenerate ones, whose elements are called **thin triangles.** There is a model structure on the category $\mathsf{sSet}^{\mathrm{sc}}$ of scaled simplicial sets, called the bicategorical model structure [Lur09a, Theorem 4.2.7]. The bifibrant objects of this model structure are called **∞-bicategories**. A morphism of scaled simplicial sets that are ∞-bicategories are called **functors** of ∞-bicategories.

Remark A.2. By [GHL22, Theorem 5.1], the fibrant objects of the bicategorical model structure are nothing but weak ∞-bicategories in the sense of [Lur09a, Definition 4.1.1]. It follows immediately that:

(1) If $\mathcal{C}$ is an ∞-category, then $(\mathcal{C}, \mathcal{C}_2)$ is an ∞-bicategory.
(2) If $\mathcal{C}$ is an ∞-bicategory, then the simplicial subset $\mathrm{Und}\,(\mathcal{C})$ of the underlying simplicial set of $\mathcal{C}$ consisting of the simplices whose 2-simplices are all thin is an ∞-category. We refer to $\mathrm{Und}\,(\mathcal{C})$ as the **underlying ∞-category** of $\mathcal{C}$, and call the objects and morphisms of $\mathrm{Und}\,(\mathcal{C})$ as **objects** and **morphisms** of $\mathcal{C}$. A morphism of $\mathcal{C}$ is called an **equivalence** if it is an equivalence in $\mathrm{Und}\,(\mathcal{C})$.
(3) If (X, T_X) is an ∞-bicategory, then so is $(X, T_X)^{\mathrm{op}} = (X^{\mathrm{op}}, T_X)$ [GHL22, Corollary 5.5].

Remark A.3. [GHL22, Remark 1.31] The ∞-bicategorical model structure is cartesian. In other words, if $A \to B$ is a cofibration and $X \to Y$ is a fibration of the bicategorical model structure, then the induced map

$$\mathrm{Fun}^{\mathrm{sc}}\,(B, X) \to \mathrm{Fun}^{\mathrm{sc}}\,(A, X) \times_{\mathrm{Fun}^{\mathrm{sc}}(A,Y)} \mathrm{Fun}^{\mathrm{sc}}\,(B, Y)$$

is again a fibration. Here $\mathrm{Fun}^{\mathrm{sc}}\,(-,-)$ denotes the internal hom of $\mathsf{sSet}^{\mathrm{sc}}$.

Example A.4. Let $\mathcal{C}$ be an ∞-bicategory. For each object $X \in \mathcal{C}$, we define a scaled simplicial set $\mathcal{C}^{X/}$ by the fiber product

$$\mathcal{C}^{X/} = \mathrm{Fun}^{\mathrm{sc}}\,([1], \mathcal{C}) \times_{\mathrm{Fun}^{\mathrm{sc}}(\{0\},\mathcal{C})} \{X\}.$$

This is an ∞-bicategory by Remark A.3.

Remark A.5. For an ∞-category $\mathcal{C}$ and an object $X \in \mathcal{C}$, there is another ∞-category $\mathcal{C}_{X/}$ equivalent to $\mathcal{C}^{X/}$. The former usually goes under the name "slice," while the latter goes by the "fat slice" [Lan21, Definition 2.5.21]. As they are equivalent, there is no essential need to distinguish between them, but we will adhere to this notational convention.

Remark A.6. A functor $p\colon \mathcal{C} \to \mathcal{D}$ of ∞-bicategories is a bicategorical fibration (i.e., fibration in the bicategorical model structure) if and only if it has the right lifting property for the following class of maps:

(1) The class of scaled anodyne extensions [Lur09a, Definition 3.1.3]. Recall that this is generated by the following maps:
  (a) For $0 < i < n$, the inclusion

$$(\Lambda^n_i, (\Lambda^n_i)_2 \cap T) \to (\Delta^n, T),$$

  where T denotes the set of degenerate 2-simplices of Δ^n and the simplex $\Delta^{\{i-2,i,i+1\}}$.

(b) The map $\left(\Delta^4, T\right) \to \left(\Delta^4, T \cup \Delta^{\{0,3,4\}} \cup \Delta^{\{0,1,4\}}\right)$, where T is the union of the degenerate 2-simplices and the simplices $\Delta^{\{0,2,4\}}$, $\Delta^{\{1,2,3\}}$, $\Delta^{\{0,1,3\}}$, $\Delta^{\{1,3,4\}}$, $\Delta^{\{0,1,2\}}$.[3]

(c) For $n > 2$, the inclusion

$$\left(\Lambda^n_0 \amalg_{\Delta^{\{0,1\}}} \Delta^0, T\right) \to \left(\Delta^n \amalg_{\Delta^{\{0,1\}}} \Delta^0, T\right),$$

where T is the set of degenerate 2-simplices of $\Delta^n \amalg_{\Delta^{\{0,1\}}} \Delta^0$ and the image of the 2-simplex $\Delta^{\{0,1,n\}}$.

(2) The inclusion $(\{\varepsilon\}, \{\varepsilon\}) \to (J, J_2)$ for $\varepsilon \in \{0, 1\}$.

This follows from Remark A.3 and the characterization of the bicategorical model structure given in [GHL22, Corollary 6.4.].

We remark that condition (1) is automatic if $\mathcal{D}$ is the scaled nerve of an ordinary category. This is because such a scaled simplicial set has a unique filler for inner horns, and at most one filler for outer horns in dimensions higher than 2.

We will often identify ∞-bicategories with their underlying simplicial sets. This is justified by the following proposition:

Proposition A.7. *Let $\overline{X} = (X, T_X)$ be a scaled simplicial set. If $\overline{X}$ is an ∞-bicategory, then the set T_X of thin triangles is completely determined by the underlying simplicial set X.*

Proof. Consider the set T'_X of 2-simplices $\sigma \in X_2$ with the following property:

($*$) Let $n \in \{3, 4\}$, and let $\phi \colon \Lambda^n_1 \to X$ be a map of simplicial sets. Suppose that $\phi|\Delta^{\{n-2,n-1,n\}}$ factors through Δ^1 via the map carrying $n-2$ to 0 and $n-1$ and n to 1. Then ϕ extends to a map $\Delta^n \to X$.

We will show that $T'_X = T_X$. Since T'_X depends only on X, this will prove the claim.

By Remark A.2 and the definition of weak ∞-bicategories, we have $T_X \subset T'_X$. To prove the reverse inclusion, suppose we are given a 2-simplex $\sigma \in T'_X$ depicted as

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z. \end{tikzcd}$$

We must show that σ is thin. For this, we use the fact that X is an ∞-bicategory to find a thin 2-simplex τ depicted as

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "gf"'] & & z. \end{tikzcd}$$

We then use condition ($*$) for $n = 3$ (applied to $\Delta^{\{0,1,2,3\}}$ and $\Delta^{\{0,1,3,4\}}$) to construct a map

$$\psi \colon \Lambda^4_1 = \Delta^{\{0,1,2,3\}} \cup \Delta^{\{0,1,2,4\}} \cup \Delta^{\{0,1,3,4\}} \cup \Delta^{\{1,2,3,4\}} \to X$$

whose restrictions to the 2-dimensional faces are given by

$$\psi|\Delta^{\{0,1,2\}} = \sigma,\ \psi|\Delta^{\{0,1,3\}} = \tau,\ \psi|\Delta^{\{1,2,3\}} = 1_g,$$

$$\psi|\Delta^{\{0,1,3\}} = \tau,\ \psi|\Delta^{\{0,1,4\}} = \sigma,\ \psi|\Delta^{\{1,3,4\}} = 1_g,$$

---

[3] The original reference by Lurie says $T \cup \Delta^{\{0,3,4\}} \cup \Delta^{\{1,3,4\}}$ instead of $T \cup \Delta^{\{0,3,4\}} \cup \Delta^{\{0,1,4\}}$, but this is a typo, as corrected in [GHL22, Definition 1.17].

and such that $\psi|\Delta^{\{0,1,2,4\}}$ is a degeneration of σ and $\psi|\Delta^{\{1,2,3,4\}}$ is a degeneration of g. Below is a picture of ψ:

$$\begin{tikzcd}
 & & x \arrow[dll, "f"'] \arrow[d, "h"] \arrow[ddl, "gf"'] \arrow[ddr, "h"] & \\
y \arrow[rr, "g"] \arrow[dr, "g"'] & & z \arrow[dl, equal] \arrow[dr, equal] & \\
 & z \arrow[rr, equal] & & z.
\end{tikzcd}$$

Using $(*)$, we can extend ψ to a map $\overline{\psi}\colon \Delta^4 \to X$. Since (X^{op}, T_X) is an ∞-bicategory and the restriction of ψ to $\Delta^{\{0,2,4\}}, \Delta^{\{1,2,3\}}, \Delta^{\{1,3,4\}}, \Delta^{\{0,1,3\}}, \Delta^{\{2,3,4\}}$ are thin ($\psi|\Delta^{\{0,1,3\}} = \tau$ is thin, and the rest are all degenerate), the definition of ∞-bicategory shows that $\overline{\psi}|\Delta^{\{0,1,4\}} = \sigma$ is thin, as required. □

Remark A.8. In [Lur25, 01W9], Lurie introduces defines an $(\infty, 2)$-category to be a simplicial set satisfying some conditions. The underlying simplicial set of an ∞-bicategory is an $(\infty, 2)$-bicategory in this sense. The converse is expected to be true, but a proof seems to be missing in the literature [hs].

Many ∞-bicategories in this paper arise from the following "nerve" constructions:

Example A.9. Let $\mathcal{C}$ be a bicategory. The **Duskin nerve** [Lur25, 009U] of $\mathcal{C}$ is an ∞-bicategory. The Duskin nerve determines a fully faithful functor from the category of bicategories and strictly unitary lax functors into the category of simplicial sets [Lur25, 00AU]. Because of this, we often identify bicategories with their Duskin nerve.

Recollection A.10. [Lur09a, Theorem 4.2.7] Let sSet^+ denote the category of marked simplicial sets (i.e., pairs (X, E_X), where X is a simplicial set and E_X is a set of edges of X containing all the degenerate ones. The category sSet^+ admits a model structure whose bifibrant objects are the ∞-categories (quasicategories) with equivalences marked. The category $\mathsf{Cat}_{\mathsf{sSet}^+}$ of sSet^+-enriched categories admits an induced model structure, and there is a Quillen equivalence

$$\mathfrak{C}^{\mathrm{sc}}\colon \mathsf{sSet}^{\mathrm{sc}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{Cat}_{\mathsf{sSet}^+} : N^{\mathrm{sc}}.$$

The right adjoint of this adjunction is called the **scaled nerve** functor.

Definition A.11. We define the ∞-bicategory $\mathcal{C}\mathsf{at}_\infty^{(2)}$ of ∞-categories to be the scaled nerve of the sSet^+-enriched category of ∞-categories, with mapping objects given by $(\mathrm{Fun}\,(\mathcal{C}, \mathcal{D}), \{\text{equivalences}\})$. Likewise, we define the ∞-bicategory $\mathcal{B}\mathsf{i}\mathcal{C}\mathsf{at}_\infty^{(2)}$ of ∞-categories as the scaled nerve of the sSet^+-enriched category of ∞-bicategories, with mapping objects given by $(\mathrm{Und}\,(\mathrm{Fun}^{\mathrm{sc}}\,(\mathcal{C}, \mathcal{D})), \{\text{equivalences}\})$. Their underlying ∞-categories are denoted by $\mathcal{C}\mathsf{at}_\infty$ and $\mathcal{B}\mathsf{i}\mathcal{C}\mathsf{at}_\infty$.

A.2. **Mapping ∞-categories.** To each ∞-bicategory $\mathcal{C}$ and each pair of objects $X, Y \in \mathcal{C}$, we can associate an ∞-category $\mathcal{C}\,(X, Y)$, called the **mapping ∞-category**. In this subsection, we define them and give a formula of the mapping ∞-categories of the arrow ∞-bicategory of ∞-bicategories.

Definition A.12. Let $\mathcal{C}$ be an ∞-bicategory. The mapping category functor $\mathcal{C}\,(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}\mathsf{at}_\infty^{(2)}$ is defined as the composite

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to N^{\mathrm{sc}}\,(\mathcal{C}_+)^{\mathrm{op}} \times N^{\mathrm{sc}}\,(\mathcal{C}_+) \cong N^{\mathrm{sc}}\left(\mathcal{C}_+^{\mathrm{op}} \times \mathcal{C}_+\right) \xrightarrow{\mathcal{C}_+(-,-)} \mathcal{C}\mathsf{at}_\infty^{(2)},$$

where the functor $\mathcal{C}_+$ is a fibrant sSet^+-enriched category equipped with an equivalence $\mathcal{C} \to N^{\mathrm{sc}}(\mathcal{C}_+)$ of ∞-bicategories, and $\mathcal{C}_+(-,-)$ denotes the hom-functor of $\mathcal{C}_+$ (with markings dropped).

If $f\colon \mathcal{C} \to \mathcal{D}$ is a functor of ∞-bicategories, then we define a natural transformation $\alpha\colon \mathcal{C}(-,-) \to \mathcal{D}(f-,f-)$ as follows: Choose $\mathcal{C}_+$ and $\mathcal{D}_+$ so that there is a sSet^+-enriched functor $f_+\colon \mathcal{C}_+ \to \mathcal{D}_+$ rendering the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & N^{\mathrm{sc}}(\mathcal{C}_+) \arrow[d] \\
\mathcal{D} \arrow[r] & N^{\mathrm{sc}}(\mathcal{D}_+)
\end{tikzcd}$$

commutative. Then α is induced by f_+.

Definition A.13. Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-bicategories. We say that f is:

- **fully faithful** if for each pair of objects $X, Y \in \mathcal{C}$, the map
$$\mathcal{C}(X,Y) \to \mathcal{D}(fX,fY)$$
is an equivalence of ∞-categories.
- **essentially surjective** if it is essentially surjective on the level of underlying ∞-categories.

Remark A.14. [Lur09a, Remark 4.2.1] Let $\mathcal{C}$ be an ∞-bicategory and let $\mathcal{C}' \subset \mathcal{C}$ be a full sub ∞-bicategory (i.e., a full simplicial subset). Then the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$ is fully faithful.

Remark A.15. [Lur09a, (Argument of) Lemma 4.2.4] A functor of ∞-bicategories is a weak equivalence in sSet^+ if and only if it fully faithful and essentially surjective. We call such a map a **bicategorical equivalence** or an **equivalence of ∞-bicategories**.

In the main body of the paper, we will often need to identify the mapping spaces of arrow ∞-bicategories. The following proposition will be useful for this. We refer the readers to [AGS23, Theorem 4.1] and [BB24, Example A.2.6] for different flavors of similar results.

Proposition A.16.

(1) *Let $\mathcal{C}$ be an ∞-bicategory, and let $f\colon X_0 \to X_1$ and $g\colon Y_0 \to Y_1$ be morphisms in $\mathcal{C}$. The mapping category of* $\mathrm{Ar}(\mathcal{C}) = \mathrm{Fun}^{\mathrm{sc}}([1],\mathcal{C})$ *fits into the pullback square*

$$\begin{tikzcd}
\mathrm{Ar}(\mathcal{C})(f,g) \arrow[r, "\mathrm{ev}_1"] \arrow[d, "\mathrm{ev}_0"'] & \mathcal{C}(X_1,Y_1) \arrow[d, "a^*"] \\
\mathcal{C}(X_0,Y_0) \arrow[r, "b_*"'] & \mathcal{C}(X_0,Y_1)
\end{tikzcd}$$

which is natural in $f, g \in \mathrm{Ar}(\mathcal{C})$.

(2) *Let C be a fibrant sSet^+-enriched category. Call a morphism $f\colon X_0 \to X_1$ a* ***quasi-fibration*** *if for each object $C \in \mathsf{C}$, the map*
$$f_*\colon \mathsf{C}(C,X_0) \to \mathsf{C}(C,X_1)$$
is a fibration in sSet^+. The full subcategory $\mathsf{C}^{[1]}_{\mathrm{qfib}} \subset \mathsf{C}^{[1]}$ spanned by quasi-fibrations is a fibrant sSet^+-enriched category, and the functor
$$\phi\colon N^{\mathrm{sc}}\left(\mathsf{C}^{[1]}_{\mathrm{qfib}}\right) \to \mathrm{Fun}^{\mathrm{sc}}([1], N^{\mathrm{sc}}(\mathsf{C}))$$
is fully faithful.

Proof. For (1), we may assume that $\mathcal{C} = N^{\mathrm{sc}}(\mathsf{C})$ for some fibrant sSet^+-category C. Let $\mathbf{A} = (\mathsf{sSet}^+)^{\mathsf{C}^{\mathrm{op}}}$ denote the sSet^+-enriched category of sSet^+-enriched functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{sSet}^+$, equipped with the projective model structure. By the enriched Yoneda embedding, the functor $\mathsf{C} \to \mathbf{A}$ is fully faithful (i.e., induces isomorphisms between hom objects). Moreover, since C is fibrant, it takes values in the full subcategory $\mathbf{A}^\circ \subset \mathbf{A}$ of bifibrant objects. Therefore, it will suffice to prove the assertion for $N^{\mathrm{sc}}(\mathbf{A}^\circ)$. In this case, [AS23, Proposition 3.89] gives an equivalence

$$N^{\mathrm{sc}}\left((\mathbf{A}^\circ)^{[1]}\right) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{sc}}([1], N^{\mathrm{sc}}(\mathbf{A}^\circ)).$$

The mapping space of $(\mathbf{A}^\circ)^{[1]}$ from $f\colon A_0 \to A_1$ to $g\colon B_0 \to B_1$ is given by the pullback

$$\begin{tikzcd}
\mathbf{A}^{[1]}(f,g) \arrow[r, "\mathrm{ev}_1"] \arrow[d, "\mathrm{ev}_0"'] & \mathbf{A}(A_1,B_1) \arrow[d, "f^*"] \\
\mathbf{A}(A_0,B_0) \arrow[r, "g_*"'] & \mathbf{A}(A_0,B_1),
\end{tikzcd}$$

which is a homotopy pullback because f^* is a fibration. The claim readily follows.

The proof of (2) is similar to that of (1) but is more elaborate. The fibrancy of $\mathsf{C}^{[1]}_{\mathrm{qfib}}$ is clear from the definition of quasi-fibrations. To show that ϕ is fully faithful, let $\mathbf{A} = \mathrm{Fun}^+(\mathsf{C}^{\mathrm{op}}, \mathsf{sSet}^+)$ be as above, and identify C with a full sub sSet^+-category via the enriched Yoneda embedding. We will also write $\mathbf{A}^\circ \subset \mathbf{A}$ for the full sSet^+-category spanned by the fibrant-cofibrant objects.

Let us say that an object $A \in \mathbf{A}^\circ$ is **good** if for each quasi-fibration $X_0 \to X_1$ in C, the induced map $\mathbf{A}(A, X_0) \to \mathbf{A}(A, X_1)$ is a fibration. We consider the following categories:

- The full sub sSet^+-category $\mathbf{A}_{\mathrm{good}} \subset \mathbf{A}^\circ$ spanned by the good objects.
- The full sub sSet^+-category $(\mathbf{A}_{\mathrm{good}})^{[1]}_{\mathrm{qfib}} \subset (\mathbf{A}_{\mathrm{good}})^{[1]}$ spanned by the quasi-fibrations in $\mathbf{A}_{\mathrm{good}}$.

Since every object in C is good, and since every quasi-fibration in C is a quasi-fibration in $\mathbf{A}_{\mathrm{good}}$, we have the following commutative diagram:

$$\begin{tikzcd}
N^{\mathrm{sc}}(\mathsf{C}^{[1]}_{\mathrm{qfib}}) \arrow[r, "\phi"] \arrow[d, hook] & \mathrm{Fun}^{\mathrm{sc}}([1], N^{\mathrm{sc}}(\mathsf{C})) \arrow[d, hook] \\
N^{\mathrm{sc}}((\mathbf{A}_{\mathrm{good}})^{[1]}_{\mathrm{qfib}}) \arrow[r, "\psi"] & \mathrm{Fun}^{\mathrm{sc}}([1], N^{\mathrm{sc}}(\mathbf{A}^\circ))
\end{tikzcd}$$

Our goal is to show that ϕ is fully faithful. Since the vertical arrows are fully faithful, it will suffice to show that ψ is fully faithful.

Let $f\colon A_0 \to A_1$ and $g\colon B_0 \to B_1$ be arbitrary objects in $(\mathbf{A}_{\mathrm{good}})^{[1]}_{\mathrm{qfib}}$. We wish to show that the map

$$N^{\mathrm{sc}}\left((\mathbf{A}_{\mathrm{good}})^{[1]}_{\mathrm{qfib}}\right)(f,g) \to \mathrm{Fun}^{\mathrm{sc}}([1], N^{\mathrm{sc}}(\mathbf{A}^\circ))(f,g)$$

is an equivalence. For this, find cofibrant replacements $\alpha\colon f' \xrightarrow{\simeq} f$ and $\beta\colon g' \xrightarrow{\simeq} g$ in $\mathbf{A}^{[1]}$, where $\mathbf{A}^{[1]}$ is equipped with the projective model structure. (This means f' and g' are cofibrations of cofibrant objects.) Consider the following sSet^+-category $\mathbf{X}$:

- The collection of objects of $\mathbf{X}$ is $\mathrm{ob}\,(\mathbf{A}_{\mathrm{good}})^{[1]}_{\mathrm{qfib}} \amalg \{f'\} \amalg \{g'\}$.

- Mapping categories are given by

$$\mathbf{X}(a,b) = \begin{cases} \emptyset & \text{if } a \in \operatorname{ob}(\mathbf{A}_{\text{good}})^{[1]}_{\text{qfib}} \text{ and } b \in \{f'\} \amalg \{g'\}, \\ \mathbf{A}^{[1]}(a,b) & \text{otherwise.} \end{cases}$$

The maps

$$\mathbf{X}(f,g) \to \mathbf{X}(f',g) \leftarrow \mathbf{X}(f',g')$$

are equivalences in sSet^+. Also, the images of α and β in $\operatorname{Fun}^{\text{sc}}([1], N^{\text{sc}}(\mathbf{A}^\circ))$ are equivalences, too. Therefore, it suffices to show that the map

$$N^{\text{sc}}(\mathbf{X})(f',g') \to \operatorname{Fun}^{\text{sc}}([1], N^{\text{sc}}(\mathbf{X}^\circ))(f',g')$$

is an equivalence. The full sub sSet^+-category of $\mathbf{X}$ spanned by f' and g' maps fully faithfully into $(\mathbf{A}^\circ)^{[1]}$, so we are reduced to showing that the map

$$N^{\text{sc}}\left((\mathbf{A}^\circ)^{[1]}\right)(f',g') \to \operatorname{Fun}^{\text{sc}}([1], N^{\text{sc}}(\mathbf{A}^\circ))(f',g')$$

is an equivalence. This follows from [AS23, Proposition 3.89]. ☐

A.3. **Cartesian fibrations.** The straightening–unstraightening equivalence is a fundamental construction in ∞-category theory. Briefly, it gives an alternative presentation of $\mathcal{C}\mathsf{at}_\infty$-valued functors in terms of (co)cartesian fibrations, which are often easier to handle than $\mathcal{C}\mathsf{at}_\infty$-valued functors themselves. In this subsection, we revisit this equivalence in the ∞-bicategorical setting.

Definition A.17. Let $p\colon \mathcal{E} \to \mathcal{B}$ be a functor of ∞-bicategories. A morphism $f\colon X \to Y$ of $\mathcal{E}$ is said to be **p-cartesian** if for each object $E \in \mathcal{E}$, the square

$$\begin{tikzcd} \mathcal{E}(E,X) \arrow[r, "f_*"] \arrow[d] & \mathcal{E}(E,Y) \arrow[d] \\ \mathcal{B}(p(E),p(X)) \arrow[r, "p(f)_*"'] & \mathcal{B}(p(E),p(Y)) \end{tikzcd}$$

is cartesian in $\mathcal{C}\mathsf{at}_\infty$. Dually, we say that f is p-cocartesian if it is p^{op}-cocartesian.

Assume now that $\mathcal{B}$ is an ∞-*category.* We say that p is a **cartesian fibration** if it satisfies the following pair of conditions:

(1) p is a fibration in the bicategorical model structure; and
(2) For each object $X \in \mathcal{E}$ and each morphism $f\colon p(X) \to B$ in $\mathcal{B}$, there is a p-cartesian morphism $X \to Y$ lying over f.

Cocartesian fibrations are defined dually.

Remark A.18. In order to maximize efficiency, our treatment of cartesian fibrations of ∞-bicategories is deliberately incomplete. For example, we can define cartesian fibrations over an arbitrary ∞-bicategory, but we have decided not to include the definitions because there are four flavors (inner and outer (co)cartesian fibrations) instead of two.

Remark A.19. In the setting of scaled simplicial sets, Definition A.17 is not a standard definition of variations of cartesian fibrations. Proving the equivalence between our definition and a more established definition is somewhat technical and is deferred to Subsection A.4.

Example A.20. Let $\mathcal{C}$ be an ∞-bicategory, and let

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "f"'] & X_0' \arrow[d, "f'"] \\ X_1 \arrow[r] & X_1' \end{tikzcd}$$

be a diagram in $\mathrm{Und}(\mathcal{C})$. Suppose that the square is cocartesian in $\mathcal{C}$ (i.e., for each $C \in \mathcal{C}$, the functor $\mathcal{C}(-, C)$ carries the square into a pullback square of ∞-categories). Then the morphism $f \to f'$ in $\mathrm{Ar}(\mathcal{C}) = \mathrm{Fun}^{\mathrm{bi}}([1], \mathcal{C})$ is cocartesian for the projection $\mathrm{Ar}(\mathcal{C}) \to \mathrm{Fun}^{\mathrm{bi}}(\{0\}, \mathcal{C}) \cong \mathcal{C}$. This follows from Proposition A.16.

Just like cartesian fibrations of ∞-categories, there is a straightening–unstraightening equivalence for ∞-bicategorical cartesian fibrations. To state this, we introduce the following notation:

Notation A.21. Let $\mathcal{B}$ be an ∞-category. We let $\mathsf{Cart}(\mathcal{B})$ denote the following category:

- Objects are cartesian fibrations $\mathcal{E} \to \mathcal{B}$ of ∞-bicategories.
- Morphisms are commutative diagrams

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "f"] \arrow[dr] & & \mathcal{E}' \arrow[dl] \\ & \mathcal{B} & \end{tikzcd}$$

of scaled simplicial sets, where f preserves cartesian edges.

We let $\mathcal{C}\mathsf{art}(\mathcal{B})$ denote the ∞-categorical localization of $\mathsf{Cart}(\mathcal{B})$ at the maps whose underlying maps are bicategorical equivalences.

We then have the following ∞-bicategorical straightening–unstraightening equivalence:

Theorem A.22. [AS23, Theorem 3.85] *For every ∞-category $\mathcal{B}$, there is an equivalence of ∞-categories*

$$\mathrm{Fun}(\mathcal{B}^{\mathrm{op}}, \mathcal{B}\mathrm{i}\mathcal{C}\mathsf{at}_\infty) \simeq \mathcal{C}\mathsf{art}(\mathcal{B}).$$

Remark A.23. As in [**?**, Appendix A], the equivalence of Theorem A.22 is natural in $\mathcal{B} \in \mathcal{C}\mathsf{at}_\infty$.

Remark A.24. [AS23, Theorem 4.21] The equivalence of Theorem A.22 is a refinement of the classical 2-categorical Grothendieck construction. More precisely, let $2\mathsf{Cat}$ denote the category of 2-categories (i.e., Cat-enriched categories) and 2-functors. Given a category $\mathbf{B}$ and a functor $F\colon \mathbf{B}^{\mathrm{op}} \to 2\mathsf{Cat}$, we can form its Grothendieck construction $\pi\colon \int F \to \mathbf{B}$. Explicitly:

- Objects of $\int F$ are pairs (B, X).
- The mapping categories are given by

$$\left(\int F\right)((B_0, X_0), (B_1, X_1)) = \coprod_{f \in \mathbf{B}(B_0, B_1)} (FB_1)(Ff(X_0), X_1).$$

The Duskin nerve of π is a cartesian fibration, and it corresponds to the composite

$$\mathbf{B}^{\mathrm{op}} \to 2\mathsf{Cat} \xrightarrow{N^D} \mathcal{B}\mathrm{i}\mathcal{C}\mathsf{at}_\infty.$$

Variant A.25. Let $\mathbf{B}$ be a category and $F\colon \mathbf{B}^{\mathrm{op}} \to \mathsf{BiCat}$ be a functor, where BiCat denotes the category of bicategories and pseudofunctors between them. We can form the Grothendieck construction $\int F$ as in Remark A.24, which is a bicategory. The Duskin nerve of the projection $\pi\colon \int F \to \mathbf{B}$ is a cartesian fibration, and it corresponds to the composite

$$\mathbf{B}^{\mathrm{op}} \to \mathsf{BiCat} \xrightarrow{N^D} \mathcal{B}\mathrm{i}\mathcal{C}\mathsf{at}_\infty.$$

To see this, note that the strictification functor $\mathrm{st}\colon \mathsf{BiCat} \to \mathsf{2Cat}$ of [GPS95, § 4.10] gives an equivalence of bicategories

$$\int F \xrightarrow{\simeq} \int \mathrm{st} \circ F.$$

The claim then follows from this equivalence and Remark A.24 Remark A.6.

A.4. **More on cartesian fibrations.** In this subsection, we will prove that cartesian fibrations in the sense of Definition A.17 are nothing but **O2C**-fibrations (or outer 2-cartesian fibrations) in the sense of [AGS22, Definition 4.22] (Corollary A.38).

Convention A.26. For disambiguation, we make the following convention throughout this subsection: We will refer to cartesian morphisms and cartesian fibrations in the sense of Definition A.17 as **fake cartesian morphisms** and fake cartesian fibrations.

With this convention, the main result of this subsection can be stated as follows:

Proposition A.27. *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a map of ∞-bicategories, where $\mathcal{B}$ is an ∞-category. Then p is a fake cartesian fibration if and only if it is an* **O2C***-fibration.*

The rest of this subsection is devoted to the proof of Proposition A.27.
We start by recalling the definition of **O2C**-fibrations.

Definition A.28. [AGS22, Definitions 2.3 and 4.7] Let X be a scaled simplicial set, and let σ be a 2-simplex of X, depicted as

$$\begin{tikzcd}
& Y \arrow[dr, "g"] \arrow[d, Rightarrow, "\sigma"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"'] & {} & Z.
\end{tikzcd}$$

- We say that σ is **left degenerate** if $\sigma|\Delta^{\{0,1\}}$ is degenerate.
- A **left degeneration** of σ is a 2-simplex τ which admits a 3-simplex $\rho\colon \Delta^3 \to X$ depicted as

$$\begin{tikzcd}
& X \arrow[r, "f"] \arrow[drr] \arrow[d, Rightarrow, "\tau"'] & Y \arrow[dr, "g"] \arrow[d, Rightarrow, "\alpha"] & \\
X \arrow[ur, "1_X"] \arrow[rrr, "h"'] & {} & {} & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
& X \arrow[r, "f"] \arrow[d, Rightarrow, "1_f"] & Y \arrow[dr, "g"] \arrow[dl, Rightarrow, "\sigma"] & \\
X \arrow[ur, "1_X"] \arrow[urr, "f"] \arrow[rrr, "h"'] & {} & {} & Z,
\end{tikzcd}$$

  where α is thin.

Definition A.29. [AGS22, Definition 2.14] A map of scaled simplicial sets is called a **weak S-fibration** if it has the right lifting property for the scaled anodyne maps.

Definition A.30. [AGS22, Definitions 4.14] Let $p\colon \mathcal{E} \to \mathcal{B}$ be a weak **S**-fibration, where $\mathcal{B}$ is an ∞-bicategory. We say that an edge $e\colon \Delta^1 \to \mathcal{E}$ is **p-cartesian** (resp. **strongly p-cartesian**) if it satisfies the following conditions:

- Let $n \geq 2$, and consider a lifting problem

$$\begin{tikzcd}
\Delta^{\{n-1,n\}} \arrow[r] \arrow[rr, bend left, "e"] & \Lambda^n_n \arrow[r, "f"] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\
& \Delta^n \arrow[r] \arrow[ur, dashed, "\widehat{f}"] & \mathcal{B}
\end{tikzcd}$$

Suppose further that $f|\Delta^{\{0,n-1,n\}}$ is thin (resp. p-cocartesian) when $n \geq 3$. Then there is a dashed filler $\widehat{f}$ such that $\widehat{f}|\Delta^{\{0,n-1,n\}}$ is thin (resp. p-cocartesian).

Remark A.31. Let $p\colon \mathcal{E} \to \mathcal{B}$ be a bicateogrical fibration of ∞-bicategories. Then a morphism of $\mathcal{E}$ is p-cartesian if and only if it is fake p-cartesian. This follows from [GHL24, Lemma 2.4.2 and Proposition 4.2.7].

Definition A.32. Let $\mathcal{C}$ be an ∞-bicategory, and let $X, Y \in \mathcal{C}$ be objects of $\mathcal{C}$. We let $\mathrm{Hom}^R_{\mathcal{C}}(X,Y)$ denote the simplicial set whose n-simplices are the maps $\phi\colon \Delta^n \star \Delta^0 \to \mathcal{C}$ such that $\phi|\Delta^n$ and $\phi|\Delta^0$ are the constant maps at X and Y. (This is the underlying simplicial set of $\mathrm{Hom}^{\triangleright}_{\mathcal{C}}(X,Y)$ defined in [GHL22, § 2.3].)

Remark A.33. [GHL22, Corollary 2.26] Let $p\colon \mathcal{C} \to \mathcal{D}$ be a weak $\mathbf{S}$-fibration of ∞-bicategories. For every pair of objects $X, Y \in \mathcal{C}$, the induced map

$$\mathrm{Hom}^R_{\mathcal{C}}(X,Y) \to \mathrm{Hom}^R_{\mathcal{D}}(p(X), p(Y))$$

is an inner fibration of ∞-categories. In particular, $\mathrm{Hom}^R_{\mathcal{C}}(X,Y)$ is an ∞-category. Moreover, its equivalences are precisely the thin triangles.

Definition A.34. [AGS22, Definition 4.1, Proposition 4.4] Let $p\colon \mathcal{E} \to \mathcal{B}$ be a weak $\mathbf{S}$-fibration, where $\mathcal{B}$ is an ∞-bicategory. We say that a left degenerate 2-simplex σ of X is **p-cocartesian** if it is cocartesian for the map

$$\mathrm{Hom}^R_{\mathcal{C}}(X,Y) \to \mathrm{Hom}^R_{\mathcal{D}}(p(X), p(Y)).$$

A 2-simplex of X is said to be **p-cocartesian** if its left degenerations are p-cocartesian. We denote the collection of cocartesian triangles by C_X.

Definition A.35. [AGS22, Definitions 4.24] Let $p\colon \mathcal{E} \to \mathcal{B}$ be a weak $\mathbf{S}$-fibration, where $\mathcal{B}$ is an ∞-bicategory. We say that p is an **outer 2-cartesian fibration**, or an **O2C-fibration**, if it satisfies the following conditions:

(O2C-1) The map p is **locally fibered** in the following sense: For each pair of objects $X, Y \in \mathcal{E}$, the functor

$$\mathrm{Hom}^R_{\mathcal{E}}(X,Y) \to \mathrm{Hom}^R_{\mathcal{B}}(p(X), p(Y))$$

is a cocartesian fibration of ∞-categories.

(O2C-2) The map p is **functorially fibered** in the following sense: Let $0 < i < 3$, and let $\rho\colon \Delta^3 \to \mathcal{E}$ be a simplex such that $\rho|\Delta^{\{i-1,i,i+1\}}$ is thin. If $\rho|\Lambda^3_i$ carries all triangles to p-cocartesian triangles, then $\rho|\Delta^{[3]\setminus\{i\}}$ is p-cocartesian. (Somewhat informally, this says cocartesian 2-cells are stable under pasting.)

(O2C-3) Every degenerate edge of $\mathcal{E}$ is strongly p-cartesian.

(O2C-4) The map p has **enough cartesian morphisms** in the following sense: Every morphism admits a p-cartesian lift with a given target.

Having recalled the definition of **O2C**-fibrations, we get down to the proof of Proposition A.27. We need a few preliminary results.

Lemma A.36. *Let $p\colon \mathcal{C} \to \mathcal{D}$ be a weak $\mathbf{S}$-fibration of ∞-bicategories. The following conditions are equivalent.*

1. *The map p is a bicategorical fibration.*
2. *The map p induces a categorical fibration of underlying ∞-categories. Equivalently (by* [Lur09b, Corollary 2.4.6.5] *and* [Lur09a, Remark 3.1.5]*), for every object $C \in \mathcal{C}$ and every morphism $f\colon p(C) \xrightarrow{\simeq} D'$ in $\mathcal{D}$, there is an equivalence $\widetilde{f}\colon C \xrightarrow{\simeq} C'$ of $\mathcal{C}$ lying over f.*

Proof. The implication (2) $\Longrightarrow$ (1) follows from Remark A.6, because the inclusion $\{\varepsilon\} \subset J$ is a trivial cofibration in the Joyal model structure for $\varepsilon \in \{0, 1\}$.

For the converse, suppose condition (1) is satisfied. We must show that condition (2) is satisfied. By Remark A.6, it will suffice to prove the following: If $f \colon \Delta^1 \to \mathcal{D}$ is an equivalence in an ∞-category, then f extends to a map $J \to \mathcal{D}$. To see this, we note that our assumption ensures that f factors through the maximal sub Kan complex $\mathcal{D}^{\simeq}$. So we may assume that $\mathcal{D}$ is a Kan complex. In this case, the claim is trivial because $\Delta^1 \hookrightarrow J$ is an anodyne extension of simplicial sets (as it is a monomorphism and both Δ^1 and J are weakly contractible). Hence (1) $\Longrightarrow$ (2), as claimed. ☐

Proposition A.37. *Let $\mathcal{B}$ be an ∞-bicategory, and let $p \colon \mathcal{E} \to \mathcal{B}$ be a weak* **S***-fibration. If p is an* **O2C***-fibration, then it is a bicategorical fibration.*

Proof. By Lemma A.36, it will suffice to show that $\mathrm{Und}\,(p)$ is a categorical fibration. This is clear, because $\mathrm{Und}\,(p)$ is a cartesian fibration of ∞-categories. ☐

Proposition A.38. *Let $\mathcal{B}$ be an ∞-category (with all triangles scaled). A weak* **S***-fibration $p \colon \mathcal{E} \to \mathcal{B}$ is an* **O2C***-fibration if and only if it satisfies the following pair of conditions:*

(1) The map p is a bicategorical fibration.
(2) Every morphism of S admits a p-cartesian lift with a given target.

Moreover, a triangle of $\mathcal{E}$ is p-cocartesian if and only if it is thin.

Proof. The last assertion is immediate from the definitions and the "three out of four" property of thin triangles [Lur09a, Remark 3.1.4]. The "only if" part of the proposition follows from Proposition A.37. For the "if" part, suppose that p satisfies conditions (1) and (2). We must show that p satisfies conditions (O2C-1) through (O2C-4) of Definition A.35:

(O2C-1) *For each pair of objects $X, Y \in \mathcal{E}$, the functor*

$$p_{X,Y} \colon \mathrm{Hom}^R_{\mathcal{E}}(X, Y) \to \mathrm{Hom}^R_{\mathcal{B}}(p(X), p(Y))$$

is a cocartesian fibration of ∞-categories. We know that $p_{X,Y}$ is an inner fibration (Remark A.33). Moreover, $\mathrm{Hom}^R_{\mathcal{B}}(p(X), p(Y))$ is a Kan complex because $\mathcal{B}$ is an ∞-category. Thus, our task is to show that every morphism of $\mathrm{Hom}^R_{\mathcal{B}}(p(X), p(Y))$ lifts to an equivalence of $\mathrm{Hom}^R_{\mathcal{E}}(X, Y)$ with a given source. This follows from the definition of weak **S**-fibrations.

(O2C-2) *The map p is functorially fibered.* This follows from the characterization of p-cocartesian triangles as thin triangles and the "three out of four" property of thin triangles.

(O2C-3) *Every degenerate edge of $\mathcal{E}$ is strongly p-cartesian.* Note that there is no distinction between strongly p-cartesian edges and p-cartesian edges, because p-cocartesian triangles are thin. Hence, it suffices to show that every degenerate edge of $\mathcal{E}$ is p-cartesian. This is immediate from Remark A.31.

(O2C-4) *The map p has enough cartesian morphisms.* This is part of the assumption.

Thus we have shown that p satisfies conditions (O2C-1) through (O2C-4) of Definition A.35, as required. ☐

We finally arrive at the proof of Proposition A.27.

Proof of Proposition A.27. This follows from Remark A.31 and Proposition A.38. ☐

A.5. **Endomorphism ∞-category.** A monoidal category can be regarded as a bicategory with a single object. This perspective gives us a fully faithful left adjoint

$$B\colon \mathsf{MonCat} \to \mathsf{BiCat}_{[0]/}$$

with right adjoint given by $(\mathcal{C}, X) \mapsto (\mathcal{C}(X, X), \circ)$. In this subsection, we record an ∞-categorical version of this observation.

Proposition A.39. *There is an adjunction of ∞-categories*

$$\mathcal{M}\mathsf{on}\mathcal{C}\mathsf{at}_\infty \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \left(\mathcal{B}\mathsf{i}\mathcal{C}\mathsf{at}_\infty\right)_{[0]/}$$

whose left adjoint is fully faithful. The essential image consists of essentially surjective functors $[0] \to \mathcal{C}$ of ∞-bicategories.

Proof. Let $\mathrm{Alg}\left(\mathsf{sSet}^+\right)$ denote the category of monoid objects in sSet^+. By [Lur17, Proposition 4.1.8.3], $\mathrm{Alg}\left(\mathsf{sSet}^+\right)$ has a model structure whose weak equivalences and fibrations are detected by the forgetful functor $\mathrm{Alg}\left(\mathsf{sSet}^+\right) \to \mathsf{sSet}^+$. The inclusion $\mathrm{Alg}\left(\mathsf{sSet}^+\right) \hookrightarrow \left(\mathsf{Cat}_{\mathsf{sSet}^+}\right)_{[0]/}$ admits a right adjoint, which carries a pointed sSet^+-enriched category $(\mathcal{C}, X)$ to the mononid object $\mathcal{C}(X, X) \in \mathrm{Alg}\left(\mathsf{sSet}^+\right)$. The resulting adjunction

$$\mathrm{Alg}\left(\mathsf{sSet}^+\right) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \left(\mathsf{Cat}_{\mathsf{sSet}^+}\right)_{[0]/}$$

is a Quillen adjunction. Moreover, the underlying ∞-categories of $\mathrm{Alg}\left(\mathsf{sSet}^+\right)$ and $\left(\mathsf{Cat}_{\mathsf{sSet}^+}\right)_{[0]/}$ can be identified with $\mathcal{M}\mathsf{on}\mathcal{C}\mathsf{at}_\infty$ and $\left(\mathcal{B}\mathsf{i}\mathcal{C}\mathsf{at}_\infty\right)_{[0]/}$ by [Lur17, Theorem 4.1.8.4] and [Cis19, Corollary 7.6.13], respectively. We thus get an induced adjunction

$$\mathcal{M}\mathsf{on}\mathcal{C}\mathsf{at}_\infty \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \left(\mathcal{B}\mathsf{i}\mathcal{C}\mathsf{at}_\infty\right)_{[0]/}.$$

By inspection, the unit of this adjunction is an equivalence, so the left adjoint is fully faithful. The claim follows. ☐

Remark A.40. A version of the Proposition A.39 appears in [GH15, Theorem 6.3.2].

Definition A.41. Let $\mathcal{C}$ be an ∞-bicategory, and let $X \in \mathcal{C}$ be its object. We write $\mathcal{E}\mathrm{nd}_{\mathcal{C}}(X)^\circ$ for the image of $(\mathcal{C}, X) \in \left(\mathcal{B}\mathsf{i}\mathcal{C}\mathsf{at}_\infty\right)_{[0]/}$ under the functor $\left(\mathcal{B}\mathsf{i}\mathcal{C}\mathsf{at}_\infty\right)_{[0]/} \to \mathcal{M}\mathsf{on}\mathcal{C}\mathsf{at}_\infty$, and call it the **endomorphism monoidal ∞-category** of $\mathcal{C}$ at X.

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